Equilibrium Search

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Seamus D. Hogan

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CONTENTS

Abstract

This thesis is concerned with the problem of incorporating consumer search into equilibrium—in particular with the relationship between oligopoly models where consumers search and those which assume perfect information.

Two chapters consider duality in search. The conventional way of expressing a consumer's search strategy is to describe his decision variable as a function of the search cost. The dual approach is to find the search cost that leaves the consumer just indifferent between two decisions. Since cost is continuous this search-cost function is differentiable in more parameters than its inverse and so more readily yields comparative static results. Using this function it is possible to express the demand facing firms as explicit functions of the distribution of consumers search costs.

The belief is commonly expressed in the literature that adaptive search models add little insight compared to the analytically simpler "rational expectations" models. It is shown that this is not valid when considering equilibrium, as the linkage between prices and consumers' search strategies implied by rational expectations affects the reaction functions of oligopolistic firms. This point is illustrated in a separate chapter which presents a Bayesian search model.

An equilibrium model allows the number of firms to be determined endogenously, and then considers the effect that entry has on efficiency and welfare. It is shown that since the monopoly power of firms arises from the incomplete information of consumers, and since the amount of search for information is endogenous—determined in part by the number of firms—entry can reduce efficiency and increase prices. It is also shown that the entry of a specialist dealer (an arbitrager) will generally reduce efficiency, and can lead to a wider dispersion of prices as the arbitrager profits from arbitraging away the inefficiency that its own presence in the market creates.

Part A: Introduction

CHAPTER 1: THESIS OVERVIEW

The now huge literature arising from the seminal article of Stigler (1961) has two main branches. The first—what may be termed "pure search models"—investigates the information-acquiring behaviour of agents faced with uncertainty over the possibilities available in a market. The second branch considers *market* outcomes when the behaviour of firms confronted with searching consumers is also modeled.

Pure search models include the formal model of Stigler's paper, in which consumers are assumed to choose the expected-price minimizing number of firms sampled from a known distribution of prices. This literature has developed by exploring the variations in sampling technique used, the objectives sought, and initial information available to the consumer.

Although these models can explain why consumers do not necessarily buy at the lowest available price, they leave open the question of why firms choose to charge different prices. In the earliest models incorporating search into a partial equilibrium—notably Diamond (1971)—the equilibrium degenerated to a single price. Rothschild (1973) challenged economists to develop models which produce price dispersion *endogenously*. Much subsequent literature is a response to that challenge, and there are now many ingenious models of equilibrium price dispersion.

As well as explaining price dispersion, models of search in equilibrium can be useful for addressing questions about the characteristics of oligopolistic markets when there is imperfect information. Here, the existing literature is less satisfactory. A major motivation of equilibrium price-dispersion models has been to find the minimum amount of of heterogeneity that can be imposed on the parameters describing agents in a model and still allow heterogeneity in prices to be obtained endogenously. As a result, these models are really elaborate examples resting on ad-hoc assumptions; it is difficult to infer from them general properties of imperfect information. It would be interesting to know, for instance, if the equilibrium properties of search vary greatly as different assumptions are made about the search process, or if they depend mainly on the universal feature of search models—that consumers allocate scarce resources to the purchase of price information in such a way that the net benefit is maximized? To address this question, it is necessary to consider the demand functions facing firms that arise from search. Pure search models have generally been developed independently of the equilibrium literature and are rarely of a form that allows tractable demand functions to be derived, that not being their objective.

Generality is an objective of this thesis. The problems involved in deriving equilibria from search models are presented more fully in the following literature survey. Part B of this thesis (Chapters 3-6) develops pure search models where the emphasis is on deriving results relevant to equilibrium. Chapter 7, which makes up Part C, is an equilibrium model. It is used to make a comparison with perfect-information oligopoly models. In particular, the role of a specialist dealer (arbitrager) is considered. As a summary of the results, some further directions of research are suggested in Part D (Chapter 8).

Throughout the thesis, the convention is used that propositions describe the mathematical properties of models and theorems present economic results. Some of the longer mathematical derivations and proofs from Chapters 3, 4, 5, and 7, which have no independent economic interest, are placed in the four appendices.

Chapter 2: Equilibrium Search — A Survey

2.1 Introduction.

This survey is not exhaustive. Four excellent surveys of search models already exist: Rothschild (1973), Lippman and McCall (1976), McKenna (1987a), and McKenna (1987b). The aim here is to provide a context for the material in this thesis. In particular, two branches of search theory, job search and general equilibrium search, which are not directly relevant to this objective are ignored. Of course, many search models can be interpreted as either consumer or job search; the branches become distinct in equilibrium modeling where the job search literature has been concerned with macroeconomic issues such as the search-theoretic foundations of the natural rate of unemployment. Lippman and McCall's is a survey of job search.

"Equilibrium search" refers here to a class of partial equilibrium models concerning the market for a consumer good. This class has the following structure: A market for a homogeneous good is characterized by a number of firms who each post a price for that good, and by a group of consumers who initially do not know the prices posted by any particular firm. Before deciding from which firm to buy, consumers can search; that is they can sample some firms at a cost to learn the prices posted by those firms. These models are concerned with establishing what prices can be posted by firms in an equilibrium, and how these are affected by search.

The demand for each firm depends on its own price through the effect price has on individual consumers' demand, and on all firm's prices through their effect on consumers' search. The profit function for each firm j is given by

$$\Pi_j(p_j,\mathbf{p}_{-j}) \equiv p_j \cdot q_j(p_j,\mathbf{p}_{-j}) - \phi_j(q_j(p_j,\mathbf{p}_{-j})), \tag{2-1-1}$$

where $\mathbf{p}_{-j}$ is the vector of all firms' prices except firm j's, $\phi_j(q_j)$ is the cost function for firm j, and $q_j(p_j, \mathbf{p}_{-j})$ is the *expected* demand for firm j.[1] The index j can take integer values only (if there are a finite number of firms), or it can be a member of a real interval (if there is a continuum of firms).

The equilibrium definition used is Nash/Bertrand: Each firm chooses its price to maximize the profit from expected demand, taking the prices of the other firms as given. An equilibrium is then a vector of prices $\mathbf{p}^*$ such that

$$\Pi_j(p_j^*, \mathbf{p}_{-j}^*) \geq \Pi_j(p_j, \mathbf{p}_{-j}^*) \qquad \forall\ p_j,\ \forall\ j. \tag{2-1-2}$$

The model structure given in (2-1-1) and (2-1-2) is equivalent to that of *product-differentiation* models. In those models, price-setting firms can face downward-sloping demand functions, as each firm's product is differentiated from its competitors. In equilibrium-search models, the good is homogeneous, so differences in the prices revealed by search are the only thing distinguishing one firm from another for any particular consumer. The information that consumers have, however, will vary across consumers following a random search process, leading to downward-sloping demand. These models can therefore be interpreted as a variant of product differentiation.

In most product differentiation models, the demand functions are imposed rather than derived from consumer behaviour.[2] As a result, assumptions can be made about demand directly. This is not possible with equilibrium search: The

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[1] Since search is a random process, each firm's demand is generally a random variable. In all the equilibrium models surveyed here, the expected value of demand is used to give a deterministic demand function. This is simpler than assuming that firms maximize expected profit, which would generally require that the full density function of demand be derived. The expected profit from demand and the profit from expected demand will only be the same when the total-cost function is linear, or when the law of large numbers implies that demand has no variance.

[2] An exception is Hotelling-type models where the differentiation is derived from the spatial dispersion of consumers.

particular assumptions made about search place restrictions on the demand functions.

One effect of having to derive demand from consumer behaviour is that it is difficult to guarantee the existence of an equilibrium when a standard search model is used. This is one reason why the equilibrium-search models tend to be special cases: So many assumptions are required to ensure existence that few free parameters remain to be used to address general questions. If, as has been the case in most equilibrium-search models, the objective of a model is to generate price dispersion in equilibrium and so *explain* search, further restrictions are imposed on the model.

The emphasis in this thesis is on investigating the relationship between search and the demand functions facing firms, and on the similarities between search models and other forms of product differentiation. The remainder of this chapter considers further the problem of generating existence and price dispersion, and summarizes the various solutions that have been found. The choice of search strategy by consumers is important to these solutions. The next section describes the variations that exist in the pure search literature that are relevant to equilibrium. Section 2.3 then considers existence, and Section 2.4, price dispersion. Finally, Section 2.5 outlines how the remaining chapters of the thesis fit into this literature and relate to the points made here.

2.2 Pure Search Models.

This section describes the original Stigler (1961) search model, and then presents other models as variations on it.

A. Stigler's Model.

A consumer wishes to buy one unit of a homogeneous good. He faces a continuum of prices described by the density function f, defined over some range of prices. The consumer knows f but not the prices charged by particular firms. He views any price as a random variable with f(p) as its density. Search is the process of realizing a value for this random variable. Stigler allowed only *fixed-sample-size* (FSS) search; that is, the consumer decides in advance how many firms to search, and then purchases at the lowest price sampled.

The consumer has a unit search cost for each firm sampled of c. Let g_n be the probability density of the sample-minimum price resulting from a sample size of n. The expected total cost to the consumer is then

$$\int p \cdot g_n(p)\ dp - nc. \tag{2-2-1}$$

where $\int p \cdot g_n(p)\ dp$ is the expected minimum cost of the purchase, and nc is the cost of acquiring quotations. The consumer is assumed to minimize this value. Stigler showed that $\int p \cdot g_n(p)\ dp$ is a concave function in n. The solution to the consumer's problem is then to choose the largest sample size such that the unit search cost does not exceed the marginal benefit, $\int p \cdot \left(g_n(p) - g_{n+1}(p)\right)\ dp$.

Alternatives to all the major assumptions in this model now exist in the search literature. The remainder of this section will present a taxonomy of these alternatives rather than of specific models.

B. Assumptions about Sampling Technique.

An early reaction to Stigler's model (for example, in McCall (1970) and Nelson (1970)) was that, by assuming FSS search, only sub-optimal behaviour by consumers was considered. The optimal strategy, it was claimed, is *sequential* search, a strategy in which consumers reconsider whether to continue search after every price quotation taken.[3] This allows consumers who receive a low

[3] The analysis of sequential sampling has a long history in statistical decision theory (see De Groot, Chapter 12.1, for a survey of some of this

quotation early to avoid wasteful additional search.

With sequential search, the consumer's problem is to calculate an *optimal stopping rule*. This is a function of the sequence of prices sampled that determines whether search should stop or continue after each quotation taken. The marginal benefit of taking another quotation is the expected reduction in the minimum price sampled. When the distribution of prices is known and only one purchase is to be made, the marginal benefit depends only on the minimum price already sampled. Let p_ℓ be that sample minimum. The expected reduction in the minimum price if a single additional quotation is taken is then

$$M(p_\ell) \equiv \int^{p_\ell} (p_\ell - p)\cdot f(p)\ dp. \qquad (2\text{-}2\text{-}2)$$

The simple model has the property of *stationarity* (the problem looks the same after each quotation). As a result of stationarity, it is also *myopic* (the decision to take one more quotation is independent of the possibility of future sampling). This provides the *reservation price rule*: Stop search if and only if the lowest price quotation is at or below the reservation price r, where r equates the marginal benefit with the marginal cost of a quotation. That is,

$$M(r) \equiv c. \qquad (2\text{-}2\text{-}3)$$

The alleged superiority of sequential over FSS search rests on the implicit assumption that there can be no savings from taking several price quotations at once. It is easy to find examples where such economies of scale exist. If there is a delay between seeking a quotation and receiving it—as with putting a contract to tender—and there is discounting, then there are

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material). McCall (1970) was the first to apply it to search.

obvious gains to seeking several quotations simultaneously. The decision to drive to a many-shop mall within which marginal search is almost costless can also be thought of as a form of FSS search. Morgan and Manning's (1985) model allows this generality. In it, consumers choose sample sizes sequentially; that is, at each period they must decide whether to continue search, and if so how many quotations to seek in the next period. This model allows both FSS and sequential search to emerge as special cases.

A feature of the reservation-price rule and stationarity in the simple sequential model is that consumers will always purchase from the last firm sampled. In more complicated sequential models, this may not be the case, so assumptions are required on the longevity of price offers. With *full recall*, consumers can always purchase from a firm previously sampled; with *zero recall*, they must purchase from the last firm sampled; *partial recall* describes all the possibilities between these two extremes. Less than full recall may be realistic in models where the duration of search is important—for instance with job search. In static consumer models, full recall is a natural assumption.

The distribution of prices facing consumers is usually assumed to be continuous. Carlson and McAfee (1983) modify sequential search to accommodate a discrete distribution of prices. Their motivation is to model equilibrium with a finite number of firms. When the price distribution is discrete, there arises a distinction between *with-replacement* and *without-replacement* sampling. This concerns whether, having visited a firm, a consumer is as likely to visit that firm again on further search or it is "removed from the draw". Carlson and McAfee only consider with-replacement sampling. Characterizations of both without-replacement sequential search and either case of discrete FSS search are missing from the literature. Chapters 3 and 4 fill this gap.

C. Assumptions about Consumers' Objectives.

Stigler's assumption that consumers seek to minimize expected expenditure has often been changed. Manning and Morgan (1982), in an FSS search model, have consumers maximize the expected indirect utility from price. While still addressing Stigler's question of how consumers behave when faced with price dispersion, this approach allows search theory to be merged into neo-classical consumer theory. For instance, the effect of income on the demand for price information can be easily considered in this framework. Veendorp (1984), using the expected indirect utility approach with sequential search, shows that, if there is a monetary cost to search (rather than only a utility cost), then the loss of stationarity that arises from search reducing income can prevent the existence of a reservation-price rule.

A particularly useful aspect of the indirect-utility-maximizing approach is that elastic consumer demand for the searched for good can be easily accommodated. Stiglitz (1987), in an equilibrium model, allows elastic demand this way, rather than using the conventional equilibrium assumption of inelastic demand with a *choke price*: the price at which demand discretely fall to zero.

Kohn and Shavell (1973) and Weitzman (1979) go further in widening consumers' objectives. In their very general models, consumers maximize expected utility when searching from a distribution of utilities. Search for the low prices is just one interpretation of this framework. For instance, Kohn and Shavell, by allowing the utility search cost to be negative, can model search for what Nelson (1970) terms *experience goods*: goods such as restaurants where sampling is through consumption rather than inspection, and where the utility of that consumption can more than compensate for the search cost.

One variation in the consumer objectives assumed in search models has received little attention. This is the relationship between search, purchase, and consumption in time. The experience-good interpretation of Kohn and Shavell is a rare exception to the implicit assumption that search precedes purchase which precedes consumption. Models recently addressing this aspect are Manning (1989a), where consumers consume while searching, and Manning (1989b), where, even if consumption is postponed, consumers are allowed to spread risk by purchasing some units of a good before terminating sequential search. Search models of this form will be required if the equilibrium-search literature is to consider issues such as reputation where time is implicit.

D. Assumptions about the Information Known to Consumers.

Perhaps the strangest assumption used in search models is that of *rational expectations:* the assumption that consumers know the distribution of prices of all firms without knowing the prices charged by specific firms. An alternative approach is *adaptive search*, in which sequentially searching consumers update their prior beliefs about the distribution of prices as search reveals information. Adaptive search is considered in McCall (1970) and Pratt, Wise, and Zeckhauser (1979). Other adaptive models have been developed by Axell (1974), Rothschild (1974), Kohn and Shavell (1974), Rosenfield and Shapiro (1981), and Morgan (1985).

There is a trade-off between the generality of adaptive search and the simplicity of rational expectations search. In the context of pure search models, the consensus has been that adaptive search models add too little insight to justify their complexity; rational expectations models are just Bayesean models with degenerate priors. In the context of equilibrium, however, the distinction has to be made between a model where consumers have rational expectations and one where there is no updating of priors. The former implies the latter but with the additional feature that consumers' priors are *correct.* This implies that if a firm changes its price, knowledge of that can affect consumers' search strategies. The demand functions of firms then reflect that each can affect the behaviour of consumers *yet to sample it.* The importance of the rational-expectations assumption in equilibrium search models has not previously been considered. It is the subject of Chapter 6.

2.3 Existence of Equilibrium.

In product differentiation models which have the structure described by equations (2-1-1) and (2-1-2), a standard assumption is that the demand functions are continuously differentiable. With some other restrictions, this assumption is used to guarantee continuous best-reply correspondences, and hence the existence of an equilibrium. When demand functions are derived from search, each firm's demand is, in many cases, discontinuous at the prices of other firms. This implies discontinuous best-reply correspondences and resulting problems for existence that are not easily assumed away.

The demand discontinuity in equilibrium-search models is a result of the assumption that firms are price setters. In the simple Bertrand model, each firm's demand function is discontinuous at the prices charged by other firms; as long as price exceeds marginal cost, each firm has an incentive to undercut its rivals. This competition for customers forces price down to marginal cost, even with only two firms.

An appeal of the differentiated product approach is that it combines the continuity of Cournot, quantity-setting behaviour with the realism of price setting. By its nature, search implies price setting, since it is for previously posted prices that consumers search. Information, however, only differentiates firms which have not been sampled (whose prices are therefore perceived as random variables) from those who have been sampled. Since the searched-for good is homogeneous, the products of two firms which have both been sampled by some consumer appear identical to that consumer, who therefore buys at the firm with the lowest price. This means that the demand discontinuities of the Bertrand model are likely to be present if there is a positive probability that two firms are both sampled by some mass of consumers. Unlike the Bertrand case, however, the competitive solution will not be an equilibrium since incomplete search by consumers also gives firms some monopoly power.

This feature, that the products of any two firms cease to be differentiated to a consumer who samples both, is common to all equilibrium-search models. Models can, however, be set up so that it does not result in discontinuous demand functions, or, if it does, that existence can still be ensured for at least a range of parameters. Four approaches have been used in the literature to generate existence of equilibrium.

First, even if some consumers are indifferent between two firms charging the same price, that price can still maximize profit if the demand function of each firm never lies above the average cost curve. If one of the two firms then lowered its price slightly, there would be a quantum increase in its demand but not in its profit. This is the case in the model of Salop and Stiglitz (1977),[4] where free entry ensures price equals average cost for all firms. They derive a two-price equilibrium for some values of parameters. The lower of the two prices is at the minimum of the U-shaped average cost curve. A demand discontinuity at the higher price is avoided because of the form of search assumed. In this special case of FSS search, consumers can either sample all firms' prices (for example, by buying a newspaper) and then buy at the lowest price, or they buy from a randomly chosen firm. As a result, no consumer samples more than one of the higher priced stores without also

[4] This model has been generalized by Braverman (1980).

sampling a lower price.

Second, if there is an infinity of firms with prices dispersed along a continuum with no mass points, there is zero probability of two identical prices being sampled by any consumer. Demand functions can therefore be continuous. This is the approach used by Wilde and Schwartz (1979), Sadanand and Wilde (1982), and Burdett and Judd (1983). They are formally similar to Butters' (1977) model which is not principally about search, but is very similar to FSS search in the way price information is disseminated (firms randomly issue advertisements to consumers, who purchase at the lowest price of these).

Third, if sequentially searching-consumers never use recall, then they are never indifferent between two prices. But zero recall does not have to be imposed: As noted in the previous section, it is a property of the basic sequential-search model that consumers purchase from the first firm visited that has a price lower than their reservation price, and so they are never indifferent between two prices. This practical attribute of sequential search perhaps explains its prevalence in equilibrium models: Axell (1977), Reinganum (1979), Carlson and McAfee (1983), Rob (1985), and Stiglitz (1987), all employ sequential search.

If there are no pure-strategy equilibria, then it is sensible to look for mixed-strategy equilibria. This is the final way that has been found to deal with the problem of demonstrating existence. Shilony (1977) derives mixed-strategy results in a model that can be interpreted as a Hotelling-style locational oligopoly or as a search based model almost equivalent to Salop and Stiglitz (1977). Varian (1980) develops the Shilony model further, giving the economic interpretation of the mixed strategy as price dispersion over time due to *sales*. Salop and Stiglitz (1982) also model sales with a mixed-strategy equilibrium.[5]

The use of mixed strategies in the Varian, and Salop and Stiglitz (1982) papers is a nice way of modeling price dispersion by firms across time. For static equilibria, the pure-strategy existence solutions are less satisfying as they lack robustness. For instance, the Salop and Stigliz (1977)/Braverman solution is very neat but it requires an *ad hoc* search model not drawn from the pure-search literature and not based on consumer behaviour. The models employing FSS search among an infinity of firms are preferable in this respect, and it does follow the tradition of pure-search models to assume a continuum of prices. Having an infinity of firms, however, is unfortunate for two reasons: First, continuous distributions are useful if they are the limiting cases of discrete distributions, but they have no interpretation on their own (there not being any market with an infinity of firms). Second, some of the interesting questions raised by price dispersion models, which are oligopolistic in structure, concern the effects on the market from changes in the number of firms. The requirement that there be an infinity of firms precludes these questions from being addressed.

An appealing generalization, applicable to the other continua models, is offered by Butters. He shows that, with a finite number of firms, there always exists an ε-equilibrium, where ε, the minimum profit required to induce a firm to change price, can be regarded as a proxy for adjustment costs. As the number of firms tends to infinity, ε tends to zero. This provides a limit interpretation of continuous models, but still precludes the use of firm numbers as a variable for comparative static analysis.

[5] Mixed strategies are not required for equilibrium in this Salop and Stigliz model. With an infinity of firms, pure strategy equilibria exist; the distribution of prices is the same as each firm's probability distribution in the mixed strategy. It is the mixed strategy, however, that has the interpretation of sales.

The third way of guaranteeing existence—using the no-recall property of sequential search—offers the most elegant solution to the problem. Again, however, existence is not robust: No-use-of-recall is a result of stationarity; it does not necessarily survive the introduction of income effects or adaptive learning. Also, without-replacement sampling (despite retaining no recall) troubles existence for other reasons. This is shown in Chapter 4.

2.4 Price Dispersion.

As noted in the previous section, mixed strategy solutions are one way of modeling dispersed-price equilibria. Generating price dispersion from pure strategies requires more work. The early models placing search in an equilibrium setting—for instance, Diamond (1971), Fisher (1970), (1972), (1973), and Hey (1974)—resulted in degeneration to a single-price equilibrium.

The reason for degeneration is that search is too successful in achieving its aim of avoiding high-priced firms. To illustrate, consider a simple FSS search model where identical consumers search for one unit of a good from a density of prices f(p). If this density is sufficiently spread to induce consumers to sample more than one firm, then there results a distribution of sample minimum prices, $g_n(p)$, which has the same support as f(p) but is stochastically dominated by it. The expected demand for firms charging the highest price is zero, so charging that price cannot be equilibrium behaviour. If, however, the density of prices is so narrow that consumers sample only one firm—resulting in identical price-offer and sample-minimum distributions—then each firm is a perfect monopolist. Now it is the lowest prices that cannot be explained. This is the logic behind the Diamond (1971), result where the only equilibrium entails every firm charging the monopoly price.

To have price dispersion in equilibrium, the assumptions about consumers must produce elastic demand functions for firms, that have the property that these yield profits at both high and low prices. This is necessary but not sufficient to explain heterogeneous price setting by firms; the assumptions made about firms are also important. This section considers both sides of the market, and finishes with a comment on the role that heterogeneity of consumers and firms plays in these models.

A. Consumers.

The simplest way of achieving elastic demand for firms is to allow each consumer to have elastic demand. This is the approach of Reinganum (1979). In her model, the equilibrium price distribution has all prices at or below the reservation price of sequentially searching consumers. As a result, all consumers buy at the first firm visited and so each firm is a perfect monopolist. It is the elastic consumer demand which provides an incentive for firms to charge below the reservation price.

Despite the consistency of elastic demand with neo-classical theory, all but Reinganum and Stiglitz (1987) of the equilibrium models cited in the previous section have eschewed this approach and assumed inelastic consumer demand. In each of these models, some, but not all consumers, sample only one firm. The consumers who sample only one firm ensure some demand for the highest priced firms; the other consumers provide the additional demand that justifies some firms' charging of lower prices. Heterogeneity in consumers' sampling behaviour has been achieved in three ways: assumed exogenously; derived endogenously as a result of *ex-ante* heterogeneity in other consumer parameters; and derived endogenously from *ex-post* heterogeneity that can result from the randomness of search.

Exogenous sampling differences is assumed by Wilde and Schwartz (1979), and in a more general version of that model, by Sadanand and Wilde (1982). Their consumers employ FSS search, but the sample sizes are not derived from beliefs about the price distribution. Rather, a certain proportion of consumers hate searching and so choose a sample size of one; the remainder love searching and choose some constant sample size exceeding one.

Heterogeneity in consumer parameters is employed by Axell (1977), Salop and Stiglitz (1977), Pratt, Wise, and Zeckhauser (1979), Braverman (1980), Von Zur Muehlen (1980), Carlson and McAfee (1983), Rob (1985), and Stiglitz (1987).[6] In each case, the consumers differ only in their search costs but this can be interpreted several ways. For instance, Salop and Stiglitz (1977) motivate search cost differences as reflecting differences in consumers' "rationality"; Gabszewicz and Garella (1985) in a degenerate-price model assume search costs to be a linear function of consumers' location along the real line.

In Burdett and Judd (1983), *ex-post* heterogeneity in consumer sampling can arise even with *ex-ante* homogeneity when, in the only dispersed price equilibrium, identical FSS searching consumers are indifferent between sampling one or two firms. This model, however, still imposes a form of heterogeneity in its assumption that consumers do choose differently when faced with equally appealing options. This difference can be reinterpreted as being based on a difference in search costs. This has the advantage of converting a knife-edge result into a robust equilibrium. Albrecht, Axell and Lang (1986) have a more appealing way of generating sampling differences *ex-post*. In their general-equilibrium model, consumers search simultaneously in a factor market and a goods market for wage/price offers. There is no recall, and search must stop in both markets simultaneously. Identical consumers will then react to differently to the same sampled price in the goods market

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[6] Von Zur Muehlen also assumes elastic consumer demand, but it is the consumer heterogeneity that is important in generating a dispersed price equilibrium in his model.

because their concurrent wage offers differ, and *vice-versa*. Anglin (1989) has a similar model, where consumers search concurrently for two goods. Consumers also differ in their search costs in that model, so there are two sources of heterogeneity.

B. Firms.

As with consumer heterogeneity, differences in firms' behaviour can be achieved through heterogeneity in parameters, or generated from within the model. The first approach is that used by Reinganum (1979), and Carlson and McAfee (1983) who both assume firms have differing cost functions. Other ways of imposing heterogeneity, such as allowing different probabilities between firms of being sampled, remain to be modeled.

Price dispersion is possible with identical firms if each has a U-shaped average cost curve, and there is free entry, as in Salop and Stigliz (1977), Braverman (1980), Wilde and Schwartz (1979), Sadanand and Wilde (1982) and Von Zur Muehlen (1982).

An alternative approach, used by Axell (1977), Burdett and Judd (1982), Rob (1985), and Stiglitz (1987), has all firms earning the same amount of monopoly profit. These four models all have an infinity of firms and the equilibrium is a price distribution which, when merged with consumers' search strategies, produces unitary elastic demand functions from which the resulting set of profit-maximizing prices is that price distribution. The infinite number of firms is important as it ensures that no deviating firm has the power to influence consumers' search behaviour and hence the profit function. For instance, Rogerson (1987) shows that Rob's model fails to produce price dispersion when there is a finite number of firms no matter how large. This is another example of why it is unsatisfactory to model equilibrium with an infinity of firms.

Of course, if there is a finite number of firms, there may be Nash equilibria where identical firms earn non-identical amounts of profit. Stiglitz (1987) gives a numerical example of a dispersed-price equilibrium in the case where there are three firms and consumers search sequentially without replacement.

C. The Role of Heterogeneity.

With the exception of the Albrecht, Axell and Lang paper, all these price-dispersion models have required heterogeneity among either consumers or firms. The Carlson and McAfee model, which is the most robust in the sense of allowing the most comparative static analysis, requires both consumers and firms to be non-identical. Imposing heterogeneity has been regarded by many authors as unsatisfactory, but the assumptions for both consumers and firms can be justified on grounds other than analytical convenience.

First, allowing differences in consumers may be an "explanation by taste", but it is consistent with differentiated product models, which formally differ from Bertrand models only when consumers are not identical. For instance, Hotelling-type differentiation-by-location models are only interesting if consumers are also distributed locationally. A lot of the insight about the role of information in oligopoly comes from comparing its results to the other types of product differentiation. Since it is consistent with those models, the assumption of consumer heterogeneity in equilibrium-search models is desirable rather than a weakness.

Allowing differences in firms' cost functions can also easily be justified. The conventional textbook justification for assuming identical costs is not that it is either an empirical observation or a useful abstraction from reality, but rather that is is a *result* of perfect competition that firms can only remain in the market if they have access to the most efficient technology. When there is imperfect competition, it is an interesting question how much technological inefficiency the market can sustain.

2.5 Relation of the Literature to the Remaining Chapters.

In concluding his survey of search models, Rothschild commented:

> *"By now it should be clear that it is possible to build logically consistent models of markets which depart in one way or another from the classical standards of perfection. In some ways it is almost too easy. The number and variety of different models make it hard to draw any firm conclusions about the consequences of different types of market failure, and it is, after all, conclusions of this sort which make the game worth the candle."*
>
> Rothschild (1973, p 1303)

The previous two sections demonstrate that a similar comment can be applied to the search-based equilibrium price dispersion models that were inspired by Rothschild's survey: The problems of generating existence and price dispersion can be overcome, but, again, the specific assumptions employed to do so produce very specific results. There is a need for equilibrium models that can be applied to a variety of situations.

This thesis works in that direction. The connecting theme of the remaining chapters is the similarity, described in Section 2.1, between equilibrium-search models and other types of product-differentiation models, and the fact that the properties specific to the former come from the restrictions search places on the demand functions facing firms.

The pure-search models of Part B consider the relationship between the assumptions made about how consumers search and the resulting demand functions, and on the implications these functions have for equilibrium. In doing so, these chapters fill in some of the gaps in the pure-search literature.

In Chapter 3, FSS search from discrete distributions of prices is analysed. The new feature here is that the sampling strategy of consumers is expressed parametrically; with-replacement and without-replacement sampling emerge as special cases from particular values of the parameters. A method is presented for deriving the demand functions facing firms as functions of the sampling strategy and a distribution of consumer search costs. The parametric form provides a means of comparison of different types of sampling strategy if these demand functions are used to derive an equilibrium. Chapter 4 considers discrete, without-replacement, sequential search (Carlson and McAfee having analysed the case of with-replacement sampling). The major result of this chapter is that, unlike with-replacement sampling (which produces analytically convenient demand functions), without-replacement sampling results in non-differentiable demand. The similarities between the results of Chapters 3 and 4 provide some interesting insights.

Chapter 5 describes an adaptive-search model where consumers have almost no information about prices before they search. Chapter 6 discusses why, contrary to the commonly stated view, the assumption that consumers have rational expectations about the price distribution is not just an innocuous special case of adaptive search when considering equilibrium. The model of Chapter 5 is used to illustrate this point.

Part C considers equilibrium directly. The single equilibrium-search there, Chapter7, draws comparisons with non-search, product-differentiation models in considering how changing the number of firms and adding a specialist dealer (arbitrager) affects welfare and efficiency in an oligopolistic market. To produce robust manipulable equilibria, heterogeneity in both consumers and firms is allowed in this model. The material of Parts C and D are not directly related. The conclusion of Part D suggests some further directions of research that could tie this material together.

Part B: Pure Search Models

Chapter 3: Search And Duality—Deriving Demand Functions From FSS Search

3.1 Introduction.

In the standard form of equilibrium search models, described in the previous chapter, search only affects the equilibrium through the demand functions facing firms. Although different assumptions made about consumers' search behaviour will produce different demand functions, and hence different equilibrium results, the procedure used to derive these functions will in many cases be the same. Much economy of effort can therefore be achieved by concentrating on the derivation of demand from search separately from specific equilibrium models.

This chapter presents a method of deriving demand functions for a broad class of fixed-sample-size (FSS) search strategies. This class includes the standard Stigler (1961) model of sampling from a continuum of prices, and various ways of sampling from discrete distributions. As well as collecting in one place a number of results that can be used in equilibrium, FSS search models, this approach allows easy comparison between the equilibrium results of different assumptions about search.

The main technique used is duality: describing consumer's search behaviour by the search cost that leaves him indifferent between two sample sizes, rather than by the optimal sample size given a particular search cost. The advantage of using duality is that it describes consumer behaviour with a continuous variable. Carlson and McAfee (1983) use the continuity of duality to derive demand functions when consumers sample sequentially from a discrete distribution of prices. By calculating the search cost required to make a particular price the reservation price, they are able to map a continuous distribution of search costs onto demand functions. A similar procedure is followed here. The continuity of the dual function is also an advantage when doing comparative static analysis of search, as it allows the variable describing consumer behaviour to be a differentiable function of the parameters in the model.

In the next section, the basic model is outlined and then duality is introduced in Section 3.3. It is shown there that, to be able to use the duality approach, it is necessary that there be diminishing returns to search. Conditions for diminishing returns are presented in Section 3.4. These are illustrated in Section 3.5 which presents examples of search behaviour that are among those described by the model. Demand functions are then derived in Section 3.6. As equilibrium models are frequently used to address the welfare implications of dispersed prices, measures of welfare appropriate to search are required. In Section 3.7, two measures of welfare, expressed in the same general notation as the rest of the chapter, are derived. In the final section, some further extensions are suggested. Proofs of two results from this chapter are found in Appendix A.

3.2 Notation and Setup of the Model.

The sampling environment facing consumers is characterized by a set of prices for a homogeneous good and a *search strategy* (to be defined). Prices are indexed by the subscript j. When the price distribution is discrete, j is an integer from the set {1, ... J}. When the distribution is continuous, j is a member of the interval [1,J]. In either case, prices are ordered from lowest to highest so that

$$i < j \Rightarrow p_i \leq p_j \qquad \forall\ i, j.$$

The price distribution is known to consumers, but, before sampling, the price charged by any firm is perceived as a random variable drawn from this distribution. The consumer will purchase some units of the good only at the minimum price sampled, and so is interested in the distribution of the sample minima rather than in the distribution of prices directly. It is the information about the sample minima distributions that we term a search strategy.

Definition 3-1:

A search strategy is a set $\{g_1, g_2, \ldots g_\infty\}$, of probability distributions over $\{p_1, \ldots p_J\}$, where $g_n(p_j)$ is the probability that p_j will be the minimum price in a sample of size n.

The definition of a search strategy as a set of probability distributions admits a wide variety of sampling behaviour. Examples of different search strategies and values of $g_n(p_j)$ for each of these are presented in Section 3.5. These include sampling with and without replacement. The latter clearly dominates the former, but, with the search strategy being considered part of the environment facing consumers, we are not concerned here with the choice of an optimal search strategy. This approach does, however, allow easy comparisons between different strategies.

G_n is the cumulative probability distribution,

$$G_n(p_j) = \sum_{i=1}^{j} g_n(p_i).$$

When the price distribution is continuous, g_n and G_n are density and distribution functions defined over the interval $[p_1, p_J]$. In general, the discrete notation is used, but, by interpreting summations as integrals, the results extend to the continuous case.

Each consumer has a search cost per firm sampled. This cost is constant except that it is assumed that there is no cost to sampling the first firm. The total expenditure on search for a consumer with search cost c sampling n firms is therefore (n-1)c. This assumption ensures that at least one firm is always sampled (and so avoids making the decision to search endogenous), but it has a useful interpretation: All purchase involves sampling the price of the firm from whom the good is bought. The cost of sampling that firm is a necessary transactions cost. By assuming the first sample to be free, only the *additional* costs that arise from introducing search into a model are being modeled as search costs.[1]

The consumer's problem given this environment is to choose the expected-utility maximizing sample size $n^* \in \mathbb{Z}_+\backslash 0$ (the set of positive integers). Where there is more than one utility maximizing n, the convention is followed that the consumer will choose n^* to be the minimum of these. A special case that is used a lot in the search literature, and that we shall often consider separately, is where consumers' demand is inelastic and independent of income. In this case maximizing expected utility gives the same result as minimizing expected expenditure.

The main concern of this chapter is with how the demand functions facing firms can be derived when all consumers in the market follow the search behaviour just described. Search mainly affects the demand functions of firms by affecting the expected number of consumers buying from particular firms. The probability that any consumer will buy from a particular firm depends on the number of firms sampled by that consumer. Let $b(n)$ be the proportion of consumers with sample size equal to n, and let $B(n)$ be the proportion with sample size greater than n so that

$$b(n) \equiv B(n-1) - B(n).$$

If consumers were identical, they would all choose the same sample size,

---

[1] This is also a common assumption in search models (e.g. see Salop (1977) and Stiglitz (1987)). An alternative way of ensuring that consumers enter the market is to assume that the marginal utility of the searched-for good tends to infinity as consumption of it tends to zero (e.g. see Manning and Morgan (1982)).

say n^*, giving $b(n^*)=1$, and $b(n)=0 \ \forall \ n \neq n^*$. As noted in Section 2.4, p 17, some kind of consumer heterogeneity is usually required to generate interesting demand functions. The standard approach of assuming that consumers differ only in their unit search costs is followed here. Let H and h be the distribution and density functions of consumers' search costs. Each value B(n) then depends on this distribution, the prices charged by firms, and the search strategy. To derive the demand functions facing firms, we need to find the relationship between $\{H, p_1,..,p_J, g_1,..,g_\infty\}$ and B(n). This relationship is derived in the next two sections using duality.

Finally in this section, we state two notational conventions used throughout the chapter. First, where any variable is parameterized by n, the standard forward difference notation is used. That is, for any variable X,

$$\Delta X_n \equiv X_{n+1} - X_n$$

and $$\Delta^2 X_n \equiv \Delta X_{n+1} - \Delta X_n.$$

Second, a circumflex "^" is used to denote expected value conditional on n,

$$\hat{X}_n \equiv E[X|n].$$

3.3 The Role of Duality.

The role of duality is illustrated here by using the example of search from a continuum of prices. This standard model was introduced in Section 2.1, p 5, but is presented here in more detail. Duality is then applied to the general model in Section 3.4.

A. Properties of the Standard Model.

Let F and f be the distribution and density functions of a continuum of prices for a good which can be sampled by a searching consumer seeking to buy one unit of the good. The distribution and density functions of sample minima are derived as follows:

The probability that all n prices sampled exceed a particular value p is

$$(1-F(p))^n.$$

The probability that at least one price is lower than p is then

$$G_n(p) \equiv 1 - (1-F(p))^n.$$

Differentiating gives

$$g_n(p) \equiv n(1-F(p))^{n-1} \cdot f(p) \tag{3-3-1}$$

A consumer with unit search cost c chooses n^* to minimize the total expected expenditure on purchase and search,

$$\int p \cdot g_n(p)\, dp + (n-1)c.$$

The marginal benefit of search is

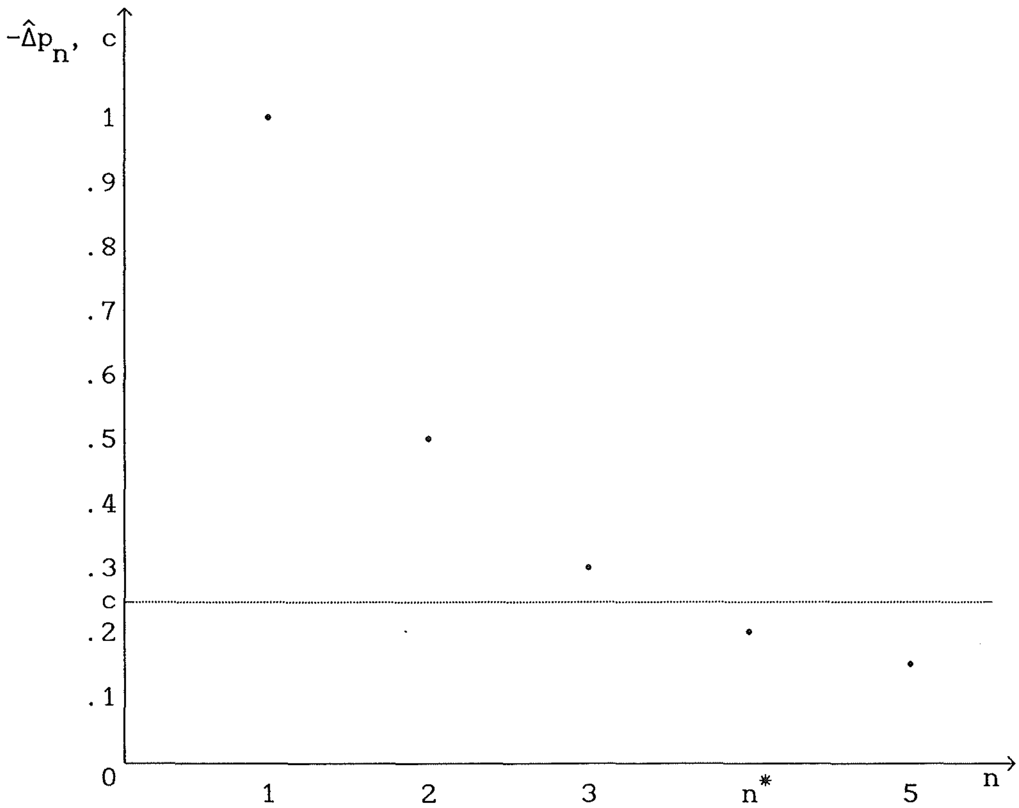
$$\begin{aligned} MB &= -\Delta\hat{p}_n \\ &= \int p \cdot g_n(p)\, dp - \int p \cdot g_{n+1}(p)\, dp \\ &= - \int p \cdot \Delta g_n(p)\, dp. \end{aligned}$$

It is readily seen by using (3-3-1) that the marginal benefit is positive but decreasing in n. The unique optimal sample size, $n^*(c)$, is then the smallest positive integer n for which the marginal benefit does not exceed the marginal cost, c.

$$n^*(c) = \min\left\{ n \,\middle|\, n \in \mathbb{Z}_+\backslash 0,\ -\Delta\hat{p}_n \le c \right\}$$

This is illustrated in Figure 3-1 for the case where f is a uniform density of prices over the interval [0, 6], and c = 0.25.

Figure 3-1.



B. Duality.

The consumer's problem described above is to select a sample of prices that would minimize total expected expenditure given the search cost c. The dual problem is to select the sample to maximize the search cost, C(n), that leaves the consumer indifferent between sampling n and n+1 firms. If C is a function and exists for all $n \in \mathbb{Z}_+\backslash 0$ it is termed the *dual function*.

In the present example, the marginal benefit of search is positive but decreasing in n. The dual function therefore exists with

$$C(n) = -\Delta\hat{p}_n.$$

The dual function provides a convenient way of relating the distribution of search costs to the distribution of sample sizes. The diminishing marginal benefit of search implies that C(n) is monotonic decreasing. Therefore, for any consumer

$$c < C(n) \Leftrightarrow n^*(c) > n. \qquad \forall\, n \tag{3-3-2}$$

It follows that the proportion of consumers whose optimal sample size is in excess of n will be the proportion whose search cost does not exceed C(n). That is,

$$B(n) = H(C(n)) \tag{3-3-3}$$

B(0), the proportion of consumers who sample at least one firm, exists and equals one. It will be convenient in the derivation of some equations later in the chapter to be able to use equation (3-3-3) when n=0. We therefore define C(0) to give H(C(0)) = 1, and, so that (3-3-2) holds, C(0) > C(1). Formally,

$$C(0) \equiv \max\left\{ \min\{c \mid H(c)=1\},\ C(1) \right\}$$

Note the importance of the dual function having a continuous range and a discrete domain. The relationship between B and H could not be stated so easily as in equation (3-3-3) using the primal—that is, a relationship between $B(n^*(c))$ and H(c)—because H is defined over continuous support while

$n^*(c)$ takes only discrete values.

The only property of the example of independent sampling from a continuum of prices that mattered in the above analysis was that the marginal benefit of search was positive and decreasing. As long as this property holds for a given search strategy, C will be a non-decreasing function defined for all positive n, and equation (3-3-2) will hold. This is why the level of generality used here can be maintained: The marginal benefit of search and sufficient conditions guaranteeing diminishing returns to search are easily stated as functions of the set of g_n; no specific search strategy needs to be appealed to. These conditions are derived in the following section under a number of assumptions about consumers' demand.

3.4 Diminishing Returns.

The characteristics of a search strategy that ensure a positive but decreasing marginal benefit of search can be described by a property of sampling distributions that is termed here *distributional dominance.*

Definition 3-2:

Let f_a and f_b be real-valued functions defined over $\{x_1, x_2, \ldots x_J\}$ with $x_1 < x_2 < \ldots < x_J$ and let F_a and F_b be the cumulative functions

$$F_a(x_j) \equiv \sum_{i=1}^{j} f_a(x_i), \quad F_b(x_j) \equiv \sum_{i=1}^{j} f_b(x_i)$$

with $F_a(x_J) = F_b(x_J)$.

Then F_a distributionally dominates F_b if and only if

$$F_a(x_j) \leq F_b(x_j) \qquad \forall\ j,$$

with strict inequality holding for at least one j.

Distributional dominance is the same concept as stochastic dominance, except that the latter is defined only over distribution functions. Here it is not required that F_a and F_b be non-decreasing or that $F_a(x_J) = F_b(x_J) = 1$. The new term is introduced so that the concept can also be applied to the first differences of distribution functions.

Assumption 3-1:

a) G_n distributionally dominates G_{n+1},

b) ΔG_{n+1} distributionally dominates ΔG_n.

The following property of distributional dominance is the basis of its role in search theory:

Lemma 3-1:

A necessary and sufficient condition for F_a to distributionally dominate F_b is

$$\sum_{j=1}^{J} p(x_j) \cdot \left(f_a(x_j) - f_b(x_j)\right) > (<) \ 0$$

for all monotonic increasing (decreasing) functions p.

Proof:

This is a well-known result for stochastic dominance.[2] To show that it also holds for distributional dominance, a proof is given in Appendix A.

□

Assumption 3-1 is sufficient to guarantee a diminishing rate of decrease in the expected minimum sampled price. When demand is inelastic, the marginal price reduction is proportional to the marginal benefit, and so Assumption 3-1

[2] For example see Lafont (1989), Chapter 2.5.

ensures the existence of the dual function C. With elastic demand, additional assumptions are required. To emphasize the role of Assumption 3-1, the case of inelastic demand is presented first.

A. Inelastic Demand.

Let the consumer have a demand, X, for the searched-for good that is independent of both price and income. The total expected cost that the consumer minimizes is

$$X \sum_{j=1}^{J} p_j \cdot g_n(p_j) + (n-1)c.$$

Proposition 3-1:

If Assumption 3-1 holds, and if consumers have inelastic demand, then the function C exists and is non-increasing with

$$C(n) = -X\Delta\hat{p}_n = -X \sum_{j=1}^{J} p_j \Delta g_n(p_j)$$

Proof:

Assumption 3-1 and Lemma 3-1 together imply that

$$\Delta\hat{p}_n = \sum_{j=1}^{J} \left(p_j \cdot g_{n+1}(p_j) - p_j \cdot g_n(p_j) \right)$$

$$= \sum_{j=1}^{J} p_j \cdot \Delta g_n(p_j) < 0$$

and $$\Delta^2\hat{p}_n = \sum_{j=1}^{J} p_j \Delta^2 g_n(p_j) > 0$$

so the expected purchase cost of X units of the good decreases at a decreasing rate in n. Therefore,

$$C(n) = -X\Delta\hat{p}_n = -X\sum_{j=1}^{J} p_j \cdot \Delta g_n(p_j) \qquad (3\text{-}4\text{-}1)$$

and $$\Delta C(n) = -X\sum_{j=1}^{J} p_j \cdot \Delta^2 g_n(p_j) \;<\; 0.$$

Also, $$n^* = \min\{\, n \mid n \in \mathbb{Z}_+ \setminus 0,\ -X\Delta\hat{p}_n \le c \,\} \qquad (3\text{-}4\text{-}2)$$

□

B. Elastic Demand with Quasilinear Utility.

The formulation with inelastic demand can be simply re-interpreted to allow for negatively sloped demand. Let consumers maximize expected utility with indirect utility functions of the form

$$U = V(p) + I - c, \qquad (3\text{-}4\text{-}3)$$

where $V'(p) \le 0$, I is income, and c can either be interpreted as a financial cost or a direct utility cost representing the irritation of search.

Proposition 3-2:

If Assumption 3-1 holds, and if consumers have quasilinear utility, then the function C exists and is non-increasing with

$$C(n) = \Delta\hat{V}_n = \sum_{j=1}^{J} V(p_j) \cdot \Delta g_n(p_j) \qquad (3\text{-}4\text{-}4)$$

Proof:

The proof follows as for Proposition 3-1, substituting $-V(p_j)$ for p_j.

□

With this formulation, the optimal sample size is given by

$$n^* = \min\{\, n \mid n \in \mathbb{Z}_+ \setminus 0,\ \Delta\hat{V}_n \le c \,\} \qquad (3\text{-}4\text{-}5)$$

□

C. General Demand.

Now let consumers' indirect utility take the general form

$$V(p, \mathbf{p}, I, n), \qquad I \equiv w - (n-1)c$$

where p is the random price of the searched for good, **p** is the vector of all other prices (known with certainty), w is the pre-search wealth and so I is income available for consumption, and n as the fourth argument represents the direct effect search has on utility.[3]

We have already seen that Assumption 3-1 implies diminishing marginal price reductions. To ensure that this also implies diminishing marginal utility, the following assumptions are required:

<u>Assumption 3-2:</u>

(a) $\dfrac{\partial^2 V}{\partial I^2} \leq 0$

(b) $\dfrac{\partial^2 V}{\partial p \partial I} \leq 0$

(c) $\left.\dfrac{\partial^2 V}{\partial n^2}\right|_I \leq 0$

(d) $\dfrac{\partial}{\partial p}\left(\left.\dfrac{\partial V}{\partial n}\right|_I\right) \geq 0$

(e) $\dfrac{\partial}{\partial I}\left(\left.\dfrac{\partial V}{\partial n}\right|_I\right) \geq 0$

<u>Proposition 3-3:</u>

If Assumptions 3-1 and 3-2 hold, then, in the problem with general utility functions, C exists, is non-increasing, and is the solution to

---

[3] A natural assumption would be that search is an irritation with an opportunity cost in time. No restriction, however, needs to be placed on the sign of $(\partial V/\partial n)|I$; "search for its own sake" is captured in this model.

$$\Delta\hat{V}_n = \sum_{j=1}^{J} V(p_j, p, w-(n-1)C(n), n)\cdot g_n(p_j) = 0 \qquad (3\text{-}4\text{-}6)$$

with $$n^* = \min\{\, n \mid n \in \mathbb{Z}_+\backslash 0,\ \Delta\hat{V}_n(p_j) \le c \,\} \qquad (3\text{-}4\text{-}7)$$

Proof:

Given in Appendix A.

□

The role of Assumption 3-2 is straightforward. There are two costs to searching: the financial cost $\left(-\frac{\partial V}{\partial I}\right)$ and the direct cost $\left(\frac{\partial V}{\partial n}\Big|_I\right)$. To demonstrate diminishing returns to search, it is required that these costs be non-decreasing in n. As n increases, income is reduced due to the financial cost, and also the expected minimum price is reduced. The second partial derivatives therefore involve n, I, and p.

This formulation of the consumer search problem in terms of the indirect utility of price is due to Manning and Morgan (1982) who considered FSS search from a continuous distribution of prices. The contribution here is to extend that formulation to allow different search strategies and search from discrete price distributions. Two differences in approach between the Manning and Morgan paper and this chapter merit comment. First, in their model n is regarded as a continuous variable, and second, they assume search has a financial cost only.

When n is continuous it can be shown that $\hat{V}_n$ is concave in n, implying diminishing returns to search. For instance, Manning and Morgan's Appendix Theorems 1 and 2 are the continuous form of Lemma 3-1 in this chapter applied to the particular functional form of g_n appropriate for sampling from a continuous distribution of prices.

Treating n as continuous also allows a convenient way of deriving and presenting comparative static results since n^* is then (generally) a differentiable function of parameters such as income, the spread of prices etc. With the exception of two Slutsky equations (which have no interpretation when n is discrete), equivalent results to those presented by Manning and Morgan can be easily derived here by considering the effect on C(n). If C(n) increases for all n when some parameter changes, then $\Delta n^* \geq 0$.

One such result is that the sample size will increase with income. Manning and Morgan comment that this result runs counter to the conventional wisdom about search. It is, however, a natural consequence of search having a financial cost only. When extending the model to allow direct effects on utility from search, Assumptions 3-2 (d) and (e) are required to ensure that C is decreasing, from which $\frac{\partial C(n)}{\partial w} > 0$ follows. It is these assumptions that violate the intuition underlying the conventional wisdom, which is that a greater income will imply a greater willingness to pay to avoid the irritation of search.

3.5 Examples of Different Search Strategies.

The definition of a search strategy as a set of probability distributions over $\{p_1, .., p_J\}$ for all n encompasses a wide variety of consumer behaviour. The purpose of this section is to illustrate a number of these variations and to check whether Assumption 3-1 holds in each case. Section 3.3 gave the example of sampling from a continuum of prices; the three strategies given in this section all involve sampling from discrete distributions.

Strategy 1: With-Replacement Sampling.

Let $F(p_j)$ be the probability, independent of n, that a firm chosen at random will charge a price not exceeding p_j. Then

$$G_n(p_j) = 1 - (1-F(p_j))^n$$

so $$\Delta G_n(p_j) = (1 - F(p_j))^n F(p_j) \qquad > 0$$

and $$\Delta^2 G_n(p_j) = -(1-F(p_j))^n \cdot F(p_j))^2 \qquad < 0$$

so Assumption 3-1 holds.

Also $$g_n(p_j) = (1-F(p_{j-1}))^n - (1-F(p_j))^n \tag{3-5-1}$$

and $$\Delta g_n(p_j) = (1-F(p_j))^n F(p_j) - (1-F(p_{j-1}))^n F(p_{j-1}) \tag{3-5-2}$$

Strategy 2: Without-Replacement Sampling.

Let there be J firms, each charging a different price and each equally likely to be sampled, so

$$g_1(p_j) = \frac{1}{J}.$$

Then $1-G_n(p_j)$ is found using the hypergeometric distribution

$$1-G_n(p_j) = \begin{cases} \dfrac{(J-j)!\,(J-n)!}{(J-j-n)!\,J!} & n \le J-j \\ 0 & \text{otherwise} \end{cases}$$

giving $$\Delta G_n(p_j) = \frac{(J-j)!}{J!}\left[\frac{(J-n)!}{(J-j-n)!} - \frac{(J-n-1)!}{(J-j-n-1)!}\right]$$

$$= \left(\frac{(J-j)!}{J!}\right)\left(\frac{(J-n-1)!}{(J-j-n-1)!}\right)\left(\frac{J-n}{J-j-n} - 1\right)$$

$$= \left(\frac{(J-j)!}{J!}\right)\left(\frac{(J-n-1)!}{(J-j-n)!}\right) j \qquad n \le J-j$$

so $$\Delta G_n(p_j) \begin{cases} > 0 & n \le J-j \\ = 0 & \text{otherwise} \end{cases}$$

$$\text{and} \qquad \Delta^2 G_n(p_j) = \left(\frac{-(J-j)!}{J!}\right)\left(\frac{(J-n-2)!}{(J-j-n)!}\right)(j-1)j \qquad n \le J-j$$

$$\text{so} \qquad \Delta^2 G_n(p_j) \begin{cases} < 0 & n \le J-j+1 \\ = 0 & \text{otherwise} \end{cases}$$

so Assumption 3-1 holds.

$$\text{Also,} \qquad g_n(p_j) = G_n(p_j) - G_n(p_{j-1})$$

$$= \begin{cases} \dfrac{(J-n)!\,(J-j)!\,n}{J!\,(J-j-n+1)!} & n \le J-j+1 \\ 0 & \text{otherwise} \end{cases} \qquad (3\text{-}5\text{-}3)$$

$$\text{and} \qquad \Delta g_n(p_j) = \begin{cases} \dfrac{(J-n-1)!\,(J-j)!}{J!\,(J-j-n+1)!}\Big((J+1)-(n+1)j\Big) & n \le J-j+1,\ n \le J-1 \\ 0 & \text{otherwise} \end{cases} \qquad (3\text{-}5\text{-}4)$$

By adding consecutive terms of $g_n(p_j)$ together, the case of several firms charging the same price can be allowed for without affecting dominance.

Strategy 3: An Example of Firm-Specific Sampling.

Let there be three firms, labelled A,B,C, posting prices $p_1 < p_2 < p_3$. As usual the consumer does not know the matching between firms and prices, but has prior beliefs. These are summarized in Table 3-1 which gives the (independent) probabilities of each firm's charging each price.

Table 3-1.

	p_1	p_2	p_3
A	.5	.2	.3
B	.3	.5	.2
C	.2	.3	.5

Since firms do not appear *ex-ante* identical to consumers in this example, the sampling distribution will depend on which stores are sampled. If consumers use without-replacement sampling in the order (ABC)—that is, firm A is sampled when the sample size is one, and firms A and B when the sample size is two—then the sampling distribution is given by Table 3-2. In this case, the required properties of distributional dominance hold.

Table 3-2. Sampling order: (ABC)

$G_n(p_j)$		p_1	p_2	p_3
n	1	.5	.7	1
	2	.8	1	1
	3	1	1	1

$\Delta G_n(p_j)$		p_1	p_2	p_3
n	1	.3	.3	0
	2	.2	0	0
	3	0	0	0

If, however, the search strategy is to sample in the order (ACB), then the sampling distribution is as given in Table 3-3, and Assumption 3-1 (b) is violated. Note that the strategy with order (ABC) is better for the consumer than that with order (ACB) in the sense that the former results in an expected minimum price that is no greater than the latter for each sample size and is less than for a sample of size 2. If the consumer were able to choose the order in which to visit firms—that is, if he were able to choose his search strategy—then (ABC) would be the optimal order and Assumption 3-1 would hold.

Table 3-3. Sampling order: (ACB)

$G_n(p_j)$		p_1	p_2	p_3
n	1	.5	.7	1
	2	.7	1	1
	3	1	1	1

$\Delta G_n(p_j)$		p_1	p_2	p_3
n	1	.2	.3	0
	2	.3	0	0
	3	0	0	0

3.6 Demand Functions.

The main objective of this chapter is to present a method of deriving demand functions that is independent of the search strategy. This is done in this section: demand functions are derived that are expressed in terms of C and $\{g_1, g_2, \ldots, g_\infty\}$. To obtain the functions for specific search strategies, the particular values for these variables, such as in the examples of the preceding section, need only be substituted in as the final stage of the derivation. As long as the dual function exists, this method is appropriate.

As noted earlier, the main effect that search has on the demand functions facing firms is through its effect on the number of consumers purchasing from particular firms. When consumer demand is inelastic, this is the only thing producing elastic demand functions for firms. When consumer demand is elastic and search has a financial cost, the firms' demand functions can be affected by income effects as well. To emphasize these separate effects, we consider the cases of inelastic consumer demand, quasilinear utility, and general utility separately.

A. Inelastic Demand.

Let q_j denote the expected demand facing firm j (where "firm j" refers to the firm charging p_j), and let there be M consumers each of whom buys X units of the good.

<u>Theorem 3-1:</u>

When consumer demand is inelastic, if there is a finite number of firms, and if $p_i \neq p_j \;\; \forall \; i \neq j$, then

$$q_j = XM\left(\sum_{n=1}^{\infty} H(C(n)) \cdot \Delta g_n(p_j) + g_1(p_j) \right)$$

and
$$\frac{\partial q_j}{\partial p_j} = -X^2 M\left(\sum_{n=1}^{\infty} \Big(h(C(n))\Big) \Big(\Delta g_n(p_j)\Big)^2 \right) < 0$$

Proof:

By the definition of b(n),

$$q_j = XM\sum_{n=1}^{\infty} b(n)\cdot g_n(p_j) \tag{3-6-1}$$

$$= XM\sum_{n=1}^{\infty}\Big[B(n-1)-B(n)\Big]\cdot g_n(p_j)$$

$$= XM\left[\sum_{n=1}^{\infty} B(n)\cdot\Delta g_n(p_j) + g_1(p_j)\right]$$

$$= XM\left[\sum_{n=1}^{\infty} H(C(n))\cdot\Delta g_n(p_j) + g_1(p_j)\right] \tag{3-6-2}$$

When $p_j \neq p_i$ for $j \neq i$, then $g_1(p_j)$ and $\Delta g_n(p_j)$ are constant and demand is differentiable with

$$\frac{\partial q_j}{\partial p_j} = XM\left[\sum_{n=1}^{\infty}\Big[h(C(n))\Big]\frac{\partial C(n)}{\partial p_j}\Delta g_n(p_j)\right]$$

Now from (3-4-1),

$$\frac{\partial C(n)}{\partial p_j} = -X\Delta g_n(p_j),$$

so

$$\frac{\partial q_j}{\partial p_j} = -X^2M\left[\sum_{n=1}^{\infty}\Big[h(C(n))\Big]\Big(\Delta g_n(p_j)\Big)^2\right] \tag{3-6-3}$$

□

The requirement in Theorem 3-1 that firm j's price be unique is important. Recall that firms are labelled according to the ordering of their prices. The functions g_1 and Δg_n are then subject to discontinuities at each of the other firm's prices and the demand functions will in general be discontinuous. This is the problem referred to in Section 2.3 p 13: Because any two firms appear otherwise identical to a consumer who has sampled both, any difference in price, no matter how slight, will differentiate them in favour of the lower-priced firm. The demand discontinuity need not arise when there are either two or an infinity of firms.

When there are only two firms, a consumer will not sample both firms if their prices are the same. This is seen in equation (3-6-2): the summation term is zero when $p_1=p_2$ since that implies that $C(n)=0\ \forall n$. Demand is then continuous as long as $g_1(p_1) = g_1(p_2)$.

As was noted in Section 2.3 p 15, when firms' prices are distributed along a continuum, the probability is zero that two firms charging the same price are sampled by any consumer, and so the discontinuity in demand disappears. Let F and f be the distribution and density functions of the continuum. The demand function of an atomistic firm can then be found as a function of the distribution.

Theorem 3-2:

When consumer demand is inelastic, and if there is an infinity of firms charging prices along a continuum, then

$$q(p) = \frac{X\Gamma}{f(p)}\left(\sum_{n=1}^{\infty} H(C(n))\cdot\Delta g_n(p) + g_1(p)\right)$$

where Γ is the ratio of consumers to firms.

Proof:

Let the number of firms, J, tend to infinity in such a way that F is the limit of the sequence $\{F^1, F^2 \ldots\}$, where each F^J is a distribution function of the prices of J firms and

$$F^J(p_j) = \frac{j}{J}.$$

The differentiable distribution, G_n, derived from F according to some search strategy, is similarly also the limit, as J tends to infinity, of a sequence $\{G_n^1, G_n^2 \ldots\}$. Let M tend to infinity in such a way that the ratio of consumers to firms, $\Gamma \equiv \frac{M}{J}$, is constant. Then from (3-6-1),

$$q_j = XM \sum_{n=1}^{\infty} b(n) \cdot \left(G_n(p_j) - G_n(p_{j-1})\right)$$

$$= XM \sum_{n=1}^{\infty} b(n) \cdot \left(G_n(p_j) - G_n(p_j-(p_j-p_{j-1}))\right) \qquad (3\text{-}6\text{-}4)$$

Now,

$$F^J(p_j) - F^J(p_{j-1}) = \frac{1}{J}$$

so
$$\frac{F(p_j) - F(p_{j-1})}{(p_j-p_{j-1})} = \frac{1}{J(p_j-p_{j-1})} \qquad (3\text{-}6\text{-}5)$$

but the sequence of F^J is defined so that as $J\ \infty$, $(p_j-p_{j-1})\ 0$. Equation (3-6-5) therefore defines a derivative giving,

$$f(p_j) = \lim_{J\ \infty} \frac{1}{J(p_j-p_{j-1})} \qquad (3\text{-}6\text{-}6)$$

Substituting (3-6-6) into (3-6-4) gives,

$$\lim_{J,M\ \infty} q_j = \frac{XM}{Jf(p_j)} \sum_{n=1}^{\infty} b(n) \cdot \left(\frac{G_n(p_j) - G_n\left(p_j - \left(\frac{1}{Jf(p_j)}\right)\right)}{\frac{1}{Jf(p_j)}} \right)$$

The bracketed term defines a derivative, so in the limit

$$q(p) = \frac{X\Gamma}{f(p)}\left(\sum_{n=1}^{\infty} b(n)\cdot g_n(p)\right)$$

$$= \frac{X\Gamma}{f(p)}\left(\sum_{n=1}^{\infty} H(C(n))\cdot\Delta g_n(p) + g_1(p)\right)$$

□

Note that if all consumers were to sample only one firm, then the market share of firm j would be $g_1(p_j)$. The summation term in equation (3-6-2) then represents the redistribution of market share from higher-priced firms to lower-priced firms that results from search.

B. Elastic Demand with Quasilinear Utility.

In this case, if the utility function takes the form of equation (3-4-3); demand is independent of the search cost and is given by Roy's identity

$$X(p) = -\frac{\partial V}{\partial p}$$

Theorem 3-3:

When consumers have quasilinear utility, if there is a finite number of firms, and if $p_i \neq p_j \quad \forall\ i \neq j$, then

$$q(p_j) = X(p)\cdot M\left(\sum_{n=1}^{\infty} H(C(n))\cdot\Delta g_n(p_j) + g_1(p_j)\right). \tag{3-6-7}$$

and

$$\frac{\partial q_j}{\partial p_j} = -M\left(X(p)^2\cdot\sum_{n=1}^{\infty}\Big(h(C(n)\Big)\Big(\Delta g_n(p_j)\Big)^2 - \frac{\partial X}{\partial p_j}\left(\sum_{n=1}^{\infty} H(C(n))\cdot\Delta g_n(p_j) + g_1(p_j)\right)\right) \tag{3-6-8}$$

Proof:

Equation (3-6-7) is the amended form of equation (3-6-2). Equation (3-6-8) follows since

$$\frac{\partial C(n)}{\partial p_j} = \frac{\partial V(p_j)}{\partial p_j} \cdot \Delta g_n(p_j) = X(p) \cdot \Delta g_n(p_j).$$

□

C. General Demand.

In the general model with a budget constraint, the demand functions are complicated by the income effects of differing c across consumers. Let $X_n(p_j, c)$ be the quantity demanded at price p_j by a consumer with income $I = w-(n-1)c$; that is, the quantity demanded by a consumer who has taken n quotations with a sample minimum p_j. Here Roy's identity gives

$$X_n(p_j, c) = -\frac{\partial V/\partial p_j}{\partial V/\partial I}.$$

Theorem 3-4:

In the problem with general utility functions, if there is a finite number of firms, and if $p_i \neq p_j \quad \forall\ i \neq j$, then

$$q_j = M \sum_{n=1}^{\infty} g_n(p_j) \int_{C(n)}^{C(n-1)} X_n(p_j, c) \cdot h(c)\, dc \tag{3-6-9}$$

and

$$\frac{\partial q_j}{\partial p_j} = M \sum_{n=1}^{\infty} g_n(p_j) \left(X_n(p_j, C(n-1)) \cdot h(C(n-1)) \cdot \frac{\partial C(n-1)}{\partial p_j} - X_n(p_j, C(n)) \cdot h(C(n)) \cdot \frac{\partial C(n)}{\partial p_j} + \int_{C(n)}^{C(n-1)} \frac{\partial X_n(p_j, c)}{\partial p_j} h(c) dc \right) \tag{3-6-10}$$

Proof:

The only difference between Equations (3-6-9) and (3-6-1) is that here demand has to be inside the integral as it depends on c. Equation (3-6-10) follows from (3-6-9).

□

It does not follow automatically from equation (3-6-10) that $\partial q_j/\partial p_j < 0$:

Although raising the price will lead to a reduction in the expected number of customers—as seen by (3-6-3)—who in turn are further up their demand curves, consumers may also have more income as a result of searching less and so demand more. This subsection, however, is included only to illustrate the complexity brought about by allowing income effects when there is a financial search cost. Clearly additional restrictions need to be made before this formulation will yield tractable demand functions.

3.7 Welfare.

The method of deriving demand functions presented in the previous section was the main concern of this chapter, but the procedure used of describing a search strategy by a set of parameters and then relying on duality has other applications. In this section, two measures of welfare for searching consumers are derived as an additional illustration of the way that keeping the description of consumers' search general can lead to both economy of effort and a more elegant presentation of results.

Welfare questions are interesting in equilibrium search models. Costly search is an activity that is not necessary for purchase of a good, and is only required when prices are believed to be dispersed. It therefore represents a loss to consumers incurred only to reduce an otherwise greater loss arising from the high prices charged by some firms. It follows that a market characterized by search will be inefficient. It is useful to have a measure welfare so that the costs of this inefficiency under different market structures can be compared. Two welfare measures are derived in this section: a dollar measure for the case of inelastic demand, and a utility measure for the case of elastic demand.

When demand is inelastic, it is convenient to measure welfare as a loss. The welfare cost, "Average Total Expenditure" (ATE) is defined as the average over all consumers of the sum of expenditure on search (ASE) and expected expenditure from purchasing the good (APE). If demand is elastic, the usual problems of defining a welfare measure when there are income effects apply. In the case where indirect utility is additively separable in income, however, Marshallian consumer surplus can be used. In this case, welfare is defined as the average over all consumers of expected utility, "Average Expected Utility" (AEU).

Theorem 3-5:

a) When consumer demand is inelastic, Average Total Expenditure is given by

$$\text{ATE} = X\hat{p}_1 - \sum_{n=1}^{\infty} \int_0^{C(n)} H(c)\, dc$$

b) When consumer demand is elastic but utility is quasilinear in Average Expected Utility is given by

$$\text{AEU} = \hat{V}_1 + \sum_{n=1}^{\infty} \int_0^{C(n)} H(c)\, dc$$

Proof:

With either utility function, consumers' average search expenditure is

$$\text{ASE} = \sum_{n=1}^{\infty} (n-1) \int_{C(n)}^{C(n-1)} c \cdot h(c)\, dc$$

$$= \sum_{n=1}^{\infty} (n-1) \left[\Big(C(n-1) \cdot H(C(n-1)) - C(n) \cdot H(C(n)) \Big) - \int_{C(n)}^{C(n-1)} H(c)\, dc \right]$$

(integrating by parts)

$$= \sum_{n=1}^{\infty} \left[C(n) \cdot H(C(n)) - (n-1) \int_{C(n)}^{C(n-1)} H(c)\, dc \right] \qquad (3\text{-}7\text{-}1)$$

The Average Purchase Expenditure for a consumer with inelastic demand is

$$APE = X \sum_{n=1}^{\infty} \left(\sum_{j=1}^{J} p_j \cdot g_n(p_j) \cdot b(n) \right) \tag{3-7-2}$$

$$= X \sum_{n=1}^{\infty} \left(\sum_{j=1}^{J} p_j \cdot \Delta g_n(p_j) \cdot H(C(n)) + \sum_{j=1}^{J} p_j \cdot g_1(p_j) \right)$$

(following the same manipulations used to derive equation (3-6-2) from equation (3-6-1)).

$$= - \sum_{n=1}^{\infty} C(n) \cdot H(C(n)) + X\hat{p}_1 \tag{3-7-3}$$

(from (3-4-1)).

Combining (3-7-1) and (3-7-3) gives

$$ATE = X\hat{p}_1 - \sum_{n=1}^{\infty} (n-1) \int_{C(n)}^{C(n-1)} H(c)\, dc$$

$$= X\hat{p}_1 - \sum_{n=1}^{\infty} (n-1) \left(\int_0^{C(n-1)} H(c)\, dc - \int_0^{C(n)} H(c)\, dc \right)$$

$$= X\hat{p}_1 - \sum_{n=1}^{\infty} \int_0^{C(n)} H(c)\, dc$$

This completes the proof for part a) of the theorem. To prove part b), note that the average per-consumer utility cost of searching is given by (3-7-1) and that the expected utility from purchase can be derived by substituting $V(p_j)$ for p_j, and $\hat{V}_1$ for $X\hat{p}_1$ in equations (3-7-2) to (3-7-3) and by using equation (3-4-4) in place of (3-4-1) in the substitution to get equation (3-7-3).

□

Note that in both cases, the first term in the welfare measure is the expected cost or utility that would result from sampling a single store; the second term represents the *gain* from search given a particular set of prices. This gain is maximized by maximizing C(n) for all n: that is, by solving the dual.

Of course, the cost or utility to a single consumer with search cost c′ can be found by substituting

$$H(c) = \begin{cases} 0 & \text{for} \quad c < c' \\ 1 & \phantom{\text{for}} \quad c \geq c' \end{cases}$$

into the formulae.

3.8 Concluding Remarks.

To get the results of this chapter, Assumption 3-1 was the only restriction that had to be placed on the search strategy used by consumers. This assumption is a sufficient condition to ensure diminishing returns to search for any price distribution, and it is a necessary condition for it to apply to *every possible* price distribution.[4] Diminishing returns to search seems an obvious property that should not need to be assumed directly, but instead be produced by more fundamental assumptions about the type of sampling allowed by consumers. Assumption 3-1 did fail to hold in the final example of Section 3.5, but this only occurred in a situation where consumers were following an inferior strategy.

The search strategy was taken as given here to emphasize the generality of the results to any strategy satisfying Assumption 3-1. An obvious extension of this work is to consider the optimal choice of search strategy. The examples of Section 3.5 suggest that there may exist some intuitive assumptions about the set from which feasible strategies are chosen such that Assumption 3-1 only fails to hold for dominated strategies.

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[4] Necessity follows directly from Lemma 3-1.

Chapter 4: Sequential Search Without Replacement

4.1 Introduction.

This chapter has the same overall objective as Chapter 3: to investigate the properties, arising from the assumptions made about consumers' search behaviour, of the demand functions facing firms. Whereas Chapter 3 emphasized generality—within the constraint that consumers use FSS search—the sequential search model considered in this chapter is specific: Sampling is without replacement from a discrete distribution of prices.

The model fills a small gap in the search literature. The optimal sequential sampling strategy for a consumer with inelastic demand searching from a continuum of prices, and the resulting demand functions have been derived by Axell (1977), Von Zur Muehlen (1980), and Rob (1985). This has been extended by Carlson and McAfee (1983) to the case where search is with replacement from a finite number of firms. Stiglitz (1987) derives properties of demand functions when consumer demand is elastic.

Two properties of sequential search are crucial in enabling these authors to model equilibrium. First, in each case there is a reservation price rule. This implies that a consumer's optimal strategy is described by a single variable; demand functions for firms can be derived by finding a mapping from consumers' search costs to the distribution of reservation prices. The second useful property is that the recall option is never exercised: Consumers always buy from the last firm visited. The problem, discussed in Section 2.3, p 13 of discontinuous demand functions is then avoided as consumers are never indifferent between *buying* from two firms.[1]

Sufficient conditions are presented in this chapter for both the reservation-price and no-recall properties to hold. The reservation-price rule

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[1] A consumer may still be indifferent between stopping and continuing search, but this does not lead to discontinuous demand.

is complicated by the fact that when sampling is without replacement the distribution of prices left to sample changes as search progresses. The reservation price then depends on the number of firms sampled, but the result is still obtained that, if there is a continuous, differentiable distribution of consumer search costs, the demand functions will be continuous. The less encouraging result is also obtained, however, that these demand functions will be characterized by non-differentiable points that produce the same problems for existence of equilibrium as discontinuities.

To derive the results on demand, the same device used in Chapter 3—duality—is employed here. In this case, the dual expresses the consumer's optimal strategy by the search cost that would make him indifferent between continuing search and stopping at a particular price. In Chapter 3, the existence of a dual function required that the optimal sample size be a monotonic function of the consumer's search cost, c. Likewise, with sequential search the existence of a dual function requires that there be a reservation price which is a monotonic function of the search cost. In both cases, this monotonicity comes from a form of diminishing returns to search. With sequential search, this implies that, the lower is the minimum price sampled, the smaller is the net gain from continuing search. Similarities between this chapter and Chapter 3 arise because of this common requirement of diminishing returns.

In the next section, the notation and assumptions of the model are given. As much as possible of the structure and notation of Chapter 3 is carried over. In Section 4.3, conditions for the existence of a reservation price are derived. Section 4.4 derives the dual function for this problem, used in Section 4.5 to produce results on the demand functions facing firms. Some brief concluding remarks are contained in Section 4.6. Many of the proofs to results from this chapter are in Appendix B.

4.2 Notation and Setup of the Model.

There are J firms posting prices $\{p_1, \ldots p_J\}$ for a homogeneous good. Firms are indexed by the ordering of their prices,

$$p_1 \le p_2 \le \ldots p_J.$$

Apart from the assumption of without-replacement sampling, the conventional set-up for sequential search is used: After each quotation, the consumer must decide whether to stop search and purchase the good at the lowest price already received (full recall), or to continue searching by sampling a firm at random. Unsampled firms appear identical to consumers so the probability that the next firm sampled charges a particular price is directly proportional to the number of unsampled firms charging that price.

Let s be the number of firms remaining to be sampled, $0 < s < J$, so that s is an index of the extent of search over time. Because there is full recall, the state of the consumer in the search process is characterized entirely by the pair (s,p_j), where p_j is the lowest price already sampled.

Search has both a financial cost and a direct utility cost. Let w be the wealth that the consumer would have remaining if he searched all J firms at the constant financial cost c, and let I be his income at point s. Then

$$I = w + sc$$

The indirect utility from purchasing the searched-for good at p_j when income is w+sc is $V(s,p_j)$. The direct disutility of taking an additional quotation is $S(s,p_j)$. Since the function S has the same domain as V and describes a marginal cost there is no loss of generality from S being additively separable from V.

There is no cost, financial or utility, to sampling the first firm. This ensures that the consumer will always start to search. Having started, the consumer's problem is to find the expected-utility maximizing stopping rule: a decision on whether to stop or continue search at every possible pair (s,p_j).

Let $U(s,p_j)$ be the expected utility of a consumer who has taken J-s quotations and for whom p_j is the lowest price sampled, assuming the optimal stopping rule is followed at that and every succeeding point. Let $Z(s,p_j)$ be the increase in expected utility from taking an additional quotation. $U(s,p_j)$ is found by the dynamic program

$$U(s,p_j) = \max \left\{ V(s,p_j),\ V(s,p_j)+Z(s,p_j) \right\} \quad j \le s+1, \qquad (4\text{-}2\text{-}1)$$

and
$$Z(s,p_j) = \frac{s-j+1}{s}U(s-1,p_j) + \frac{1}{s}\sum_{i=1}^{j-1} U(s-1,p_i) - S(s,p_j) - V(s,p_j) \quad (4\text{-}2\text{-}2)$$

where
$$U(s,p_1) \equiv V(s,p_1).$$

In equation (4-2-1), the value of continuing search is expressed as the value of stopping added to the *marginal* gain from continuing. The separate definition of a marginal benefit makes it easier to transfer notation from the general case of utility maximization to the special case, considered in the next section, of expenditure minimization.

Note that if the s firms remaining to be sampled are those with the s lowest prices, the lowest price sampled till then will be p_{s+1}. For any other combination of remaining prices, the lowest quotation will be less than this. The index of the lowest quotation, j, can therefore never exceed s+1. So that equation (4-2-2) is defined for j = s+1, however, the variable $U(s,p_{s+2})$ is defined but it has no interpretation as it always has a coefficient of zero. Likewise the convention is used throughout that

$$\sum_{i=a}^{b} (\cdot) \equiv 0 \quad \text{when} \quad a > b.$$

Where a consumer is indifferent between stopping and continuing, the convention is followed that the consumer will choose to stop. With this notation, the consumer's optimal stopping rule can be stated: *Stop search if and only if* $Z(s,p_j) \leq 0$.

4.3 Reservation Prices.

To clearly identify the source of results, we consider the increasingly general cases of inelastic demand, quasilinear utility, and general elastic demand.

A. Inelastic Demand.

Let the consumer demand one unit of the searched for good,[2] let him only have a financial search cost and be risk neutral. It is convenient in this case to consider reductions in expected expenditure rather than increases in expected utility. Let $A(s,p_j)$ be the expected cost (from purchase and *additional* search) after J-s quotations if the lowest quotation received at that point is p_j and assuming the optimal stopping policy is followed at that and every succeeding point. Then,

$$A(s,p_j) = \min\left\{ p_j,\ p_j - Z(s,p_j) \right\} \qquad j \leq s+1 \tag{4-3-1}$$

where
$$Z(s,p_j) = p_j - \left(\frac{s-j+1}{s} A(s-1,p_j) + \frac{1}{s}\sum_{i=1}^{j-1} A(s-1,p_i) + c \right) \tag{4-3-2}$$

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[2] This easily extends to inelastic demand for X units of the good: By choosing appropriate units of measurement for price, X can be arbitrarily set equal to one without loss of generality. A change in X is then equivalent to an inversely proportional change in c.

and $A(s,p_1) \equiv p_1$

$A(s,p_j)$ is the negative of a utility measure. More precisely, there exists the indirect-utility function

$$V(s,p) \equiv -p + w + cs \tag{4-3-3}$$

that describes the preferences underlying the inelastic demand, giving

$$A(s,p_j) \equiv -U(s,p_j).$$

The definitions of $Z(s,p_j)$ in (4-3-2) and (4-2-1) are therefore consistent with each other; the optimal stopping policy is still to continue search only when $Z(s,p_j)$ is positive. In this case, $Z(s,p_j)$ is the expected reduction in total expenditure from taking an additional quotation. As with $U(s,p_{s+2})$, $A(s,p_{s+2})$ is defined but has no interpretation.

For a given distribution of prices, the marginal benefit of search must be lower, the lower the minimum price already observed since the price *reduction* associated with any newly sampled price will be lower. In the case of inelastic demand, this automatically implies diminishing returns to search (and hence a reservation price) since the search cost is independent of price.

Proposition 4-1:

For each s, there is a reservation price r^s such that search will terminate if $p_j \le r^s$ and continue otherwise. That is,

$$\exists\ k \quad \text{s.t.} \quad Z(s,p_j) \le 0 \qquad\qquad \forall\ j \le k.$$

Proof:

It is sufficient to show that $Z(s,p_j)$ is non-decreasing in p_j, for all $j \le s+1$, for all s. The proof is by induction.

a) From (4-3-2)

$$Z(1,p_1) = -c$$

and $$Z(1,p_2) = p_2 - p_1 - c,$$

so $$Z(1,p_1) \le Z(1,p_2),$$

and the proposition holds for s=1.

b) Assume that $Z(s-1,p_j)$ is non-decreasing in p_j. Then,

$$A(s-1,p_{j+1}) - p_{j+1} \le A(s-1,p_j) - p_j \qquad j \le s-1 \tag{4-3-4}$$

Now $$Z(s,p_{j+1}) - Z(s,p_j)$$

$$= p_{j+1} - \left(\frac{s-j}{s}A(s-1,p_{j+1}) + \frac{1}{s}\sum_{i=1}^{j-1} A(s-1,p_i) + \frac{1}{s}A(s-1,p_j) + c\right)$$
$$- p_j + \left(\frac{s-j+1}{s}A(s-1,p_j) + \frac{1}{s}\sum_{i=1}^{j-1} A(s-1,p_i) + c\right)$$

$$= p_{j+1} - p_j + \frac{s-j}{s}\left(A(s-1,p_j) - A(s-1,p_{j+1})\right)$$

which is positive (from (4-3-4)). It follows that if the proposition holds for s-1, it holds for s.

□

Let $C(s,p_j)$ be the search cost that would leave the consumer indifferent between stopping and continuing search at the point (s,p_j)—that is, the search cost for which $Z(s,p_j) = 0$. The existence of a reservation price rule means that the consumer's optimal policy can be described by a single variable r^s. To use duality in deriving demand functions, $C(s,p_j)$ needs to be a function defined for all (s,p_j). This requires that r^s be a monotonic function of c.

Corollary:

The reservation price is non-decreasing in c.

Proof:

From equation (4-3-2), $\frac{\partial Z}{\partial c} < 0$. The result follows.

□

The diminishing returns to search that led to Proposition 4-1 reflected the fact that the sampling distribution of price *reductions* is different for different values of p_j given s. This property, of course, applies to with-replacement search as well. There is a second effect when sequential sampling is without replacement As the number of firms remaining to be sampled changes, there is a change in the sampling distribution of prices. This results in a form of increasing returns to search over time: The more firms that have already been sampled for a given minimum price, the more chance there is of sampling the lower prices on succeeding searches, and so the greater is the expected gain from search. This leads to a no-recall result as it implies that the reservation price is non-increasing as search progresses (that is, non-decreasing in s). A non-increasing reservation price is a necessary and sufficient condition for consumers to always buy from the last firm sampled.

Proposition 4-2:

The recall option will never be exercised.

Proof:

If the reservation price is non-decreasing in s, then

$$A(s-1,p_j) = p_j \Rightarrow A(s,p_j) = p_j.$$

Let p_k be the reservation price at s-1 so $Z(s-1,p_k) \leq 0$. It is sufficient to show that

$$Z(s-1,p_k) \le 0 \quad \Rightarrow \quad Z(s,p_k) \le 0.$$

The reservation price rule implies that

$$A(s,p_k) = \min\left\{ p_k, \ \frac{s-k+1}{s}p_k + \frac{1}{s}\sum_{i=1}^{k-1} p_i + c \right\}$$

$$\Rightarrow \qquad Z(s,p_k) = p_k - \left(\frac{s-k+1}{s}p_k + \frac{1}{s}\sum_{i=1}^{k-1} p_i + c\right)$$

and, by definition,

$$Z(s-1,p_k) = p_k - \left(\frac{s-k}{s-1}A(s-2,p_k) + \frac{1}{s-1}\sum_{i=1}^{k-1} A(s-2,p_i) + c\right)$$

$$= p_k - \left(\frac{s-k+1}{s}A(s-2,p_k) + \frac{1}{s}\sum_{i=1}^{k-1} A(s-2,p_i) - \frac{1}{s(s-1)}\sum_{i=1}^{k-1}\Big(A(s-2,p_k) - A(s-2,p_i)\Big) + c\right)$$

so

$$Z(s,p_k) - Z(s-1,p_k) = \frac{s-k+1}{s}\Big(A(s-2,p_k)-p_k\Big) + \frac{1}{s}\sum_{i=1}^{k-1}\Big(A(s-2,p_i)-p_i\Big) - \frac{1}{s(s-1)}\sum_{i=1}^{k-1}\Big(A(s-2,p_k) - A(s-2,p_i)\Big)$$

Now clearly,

$$A(s-2,p_k) \ge A(s-2,p_i), \qquad k > i$$

and, from (4-3-1),

$$A(s-1,p_j) \le p_j \qquad \forall j$$

so that

$$Z(s,p_k) - Z(s-1,p_k) \le 0.$$

It follows that

$$Z(s-1,p_k) \le 0 \Rightarrow Z(s,p_k) \le 0.$$

□

The consumer's optimization problem can be shown on a grid where each cell represents a pair (s,p_j). This is illustrated in Figure 4-1 where each cell contains the value $A(s,p_j)$ calculated for an example where prices are successive multiples of 12 and c = 10. The triangular shape of the grid results from the assumption of with-replacement sampling. Proposition 4-1 implies that the grid can be partitioned into two regions by a single line following the reservation prices (r^5=36, r^4=r^3=r^2=24, r^1=r^0=12), shown as a heavy border in Figure 4-1. The upper region represents the decision to continue search and the lower region the decision to stop. Proposition 4-2 shows that this line is non-increasing. The figure shows, for example, that a searcher whose lowest quotation is 48 when 3 firms remain to be sampled will choose not to buy at that price, but will continue search with expected additional expenditure of 31. In the region for which search stops, $A(s,p_j)=p_j$, so entries horizontal to each other are identical. To the right of the reservation-price border, the entries decrease from left to right, illustrating the increasing returns property mentioned above. The equality of the values for $A(s,p_j)$ above the reservation-price border, for any s, reflects the fact that the reservation price is non-increasing as s decreases. For instance, a searcher who decides to continue search when 5 firms remain unsampled will then never stop while the lowest quotation exceeds 24. Whether the first quotation was 48, 60, or 72 is irrelevant to his expected expenditure.

Figure 4-1. Table of $A(s,p_j)$ when $c = 10$.

$p_6=72$	37.9					
$p_5=60$	37.9	34.5				
$p_4=48$	37.9	34.5	31			
$p_3=36$	36	34.5	31	27		
$p_2=24$	24	24	24	24	22	
$p_1=12$	12	12	12	12	12	12
s	5	4	3	2	1	0

B. Elastic Demand with Quasilinear Utility.

It was noted earlier that a property of inelastic demand that ensures a reservation price rule is that the marginal cost of search is independent of price. If equation (4-3-3) is modified so that the consumer's indirect utility function takes the form

$$V(s,p) \equiv v(p) + w + cs,$$

then demand can be elastic and the search cost still be independent of price. Propositions 4-1 and 4-2 then extend, with the only modification to the proofs being that $-v(p)$ needs to be substituted for p_j in the expressions for $A(s,p_j)$ and $Z(S,p_j)$.

C. General Elastic Demand.

In this formulation, marginal search costs need not be independent of price. To generalize Proposition 4-1 the following additional assumptions are made.

Assumption 4-1:

a) $\dfrac{\partial^2 V}{\partial p \partial I} \le 0$

b) $\dfrac{\partial S}{\partial p} \le 0$

Proposition 4-3:

If Assumption 4-1 holds then, for each s, there is a reservation price r^s such that search will terminate if $p_j \le r^s$ and continue otherwise. That is,

$$\exists\ k \quad \text{s.t.} \quad Z(s, p_j) \le 0 \quad \forall\ j \le k.$$

Proof:

Given in Appendix B.

□

Veendorp (1985) has shown, in the case of with-replacement sampling, that the reservation price rule does not necessarily hold when income effects are allowed for. He gives a sufficient condition that the marginal utility of income falls in price. It is interesting that this assumption is also used by Manning and Morgan (1982) to ensure a unique optimal sample size when extending FSS search to allow income effects. The requirement is that the marginal cost of search be higher at lower prices. In this chapter, because a direct utility cost has been allowed in addition to the financial cost, Assumption 4-1 (b) is made. This also has a counterpart in Chapter 3—Assumption 3-2 (d).

The properties of a reservation price rule in sequential search and of a unique local optimal search size in FSS search seem unrelated but both arise from the common objective, referred to in the introduction, of showing diminishing returns to search intensity. Veendorp presents an example where a consumer will stop search at high or low prices and continue at prices in the middle because the expected utility reduction from spending income on search is great at high prices. The assumption that $\partial^2V/\partial p\partial I \le 0$ ensures that the utility loss is greater at lower prices; this supplements the property that the lower the best quote received, the lower the probability of reducing it on further search. In the Manning and Morgan model, increasing the number of firms searched reduces the expected price. In this case, $\partial^2V/\partial p\partial I \le 0$ then ensures that the indirect utility reduction from search is non-decreasing in n.

It is not, then, that a reservation price rule and a unique optimal sample size are equivalent properties in some sense, but that they depend on a common property—diminishing returns. The following no-recall result, however, relies on increasing returns as more prices are sampled. The sufficient conditions used to generalize Proposition 4-2 therefore have the reverse inequalities from the equivalent conditions in Assumption 3-2.

<u>Assumption 4-2:</u>

a) $\dfrac{\partial^2 V}{\partial I^2} \ge 0$

b) $\dfrac{\partial S}{\partial s} \ge 0$

c) $\dfrac{\partial^2 V}{\partial p \partial I} \ge 0$

Assumption 4-2 (c) contradicts the sufficient condition for the reservation-price rule to hold. The proof of the following proposition showing no recall uses both Assumption 4-2 and the reservation-price property. For there to be a non-increasing reservation price it is necessary that $|\partial^2V/\partial I\partial p|$ and c be low relative to $\partial^2V/\partial I^2$ and s. This, of course is true in the case of quasilinear utility, which is why the no-use-of-recall could be proven there.

Proposition 4-4:

If the reservation price property and Assumption 4-2 hold, then the recall option will not be exercised.

Proof:

Given in Appendix B.

□

4.4 The Dual Function.

In this and the following section, only inelastic consumer demand is considered. The basic result, that demand functions are not everywhere differentiable, carries over to the more general formulations though.

Recall that $C(s,p_j)$ is the search cost that would leave the consumer indifferent between stopping and continuing at point (s,p_j), so, from (4-3-1),

$$p_j = \frac{s-j+1}{s}A(s-1,p_j) + \frac{1}{s}\sum_{i=1}^{j-1} A(s-1,p_i) + C(s,p_j) \tag{4-4-1}$$

The first objective here is to find an expression for $C(s,p_j)$ as a function of the prices $p_1 \ldots p_j$.

The difficulty in solving in general for $C(s,p_j)$ as a function of prices is that each term $A(s,p_j)$ involves a succession of minimum functions when expanded out. This is not a problem when $j=1$ or $j=2$, however. To get a feel for the problem these two cases are considered first before a formulation for the general case is extracted. Where $C(s,p_j)$ can be written c without ambiguity, this will be done.

A. j=1.

Clearly, indifference requires $C(s,p_1) = 0 \quad \forall$ s.

B. j=2.

From (4-4-1),

$$p_2 = \frac{s-1}{s}A(s-1,p_2) + \frac{1}{s}p_1 + C(s,p_2) \tag{4-4-2}$$

$C(s,p_2)$ is defined so that continuing search at (s,p_2) is an optimal action. By Proposition 4-2, it must therefore be optimal at all (t,p_2) for $t<s$. Therefore,

$$A(t,p_2) = \frac{t-1}{t}A(t-1,p_2) + \frac{1}{t}p_1 + c \qquad 0 < t < s.$$

Equation (4-4-2) can then be expanded out

$$p_2 = \frac{s-1}{s}\left(\frac{s-2}{s-1}A(s-2,p_2) + \frac{1}{s-1}p_1 + c\right) + \frac{1}{s}p_1 + c$$

and so on until t=1, giving

$$\begin{aligned} p_2 &= s\left(\frac{1}{s}\right)p_1 + \sum_{k=0}^{s-1}\left(\frac{s-k}{s}\right)c \\ &= p_1 + \left(\frac{s+1}{2}\right)c \end{aligned}$$

so $$C(s,p_2) = \frac{2}{s+1}(p_2-p_1).$$

Note that when one firm remains to be sampled, indifference requires that the search cost be equal to the (known) reduction in price. The fact that $C(s,p_2)$ increases as s decreases is the representation in the dual function of the increasing-returns property that gave non-increasing reservation prices.

C. The general case.

When $j > 2$, for $c = C(s,p_j)$, Proposition 4-2 implies that for any $t < s$,

$$A(t,p_j) = \frac{t-j+1}{t}A(t-1,p_j) + \frac{1}{t}\sum_{i=1}^{j-1} A(t-1,p_i) + c$$

Any term, $A(s-1,p_j)$, can therefore be expanded out as a function of $A(t,p_i)$, $t < s-1$, $i<j$, as in the case where $j=2$. For $1<i<j$, the minimum function in the definition of A (equation (4-3-1)) cannot be ignored. Here again, though, Proposition 4-2 simplifies the problem. As the reservation prices are non-increasing as s decreases, the consumer's optimal policy can be described by a set of image reservation prices: values x_j defined for each p_j such that search will stop at p_j for every $s \geq x_j$. Formally,

$$x_j = \min\left\{ s \mid A(s,p_j)=p_j \right\}$$

For instance, in Figure 4-1, $x_1=0$, $x_2=2$, $x_3=5$, and x_4, x_5, x_6 are undefined. With this notation,

$$A(t,p_i) = \begin{cases} p_i & x_i \leq t \\ \frac{t-i+1}{t}A(t-1,p_i) + \frac{1}{t}\sum_{h=1}^{i-1} A(t-1,p_h) + c & x_i > t \end{cases} \qquad (4\text{-}4\text{-}3)$$

All terms $A(t,p_i)$ can then be successively expanded out as functions of x_i. Let $\mathbf{x} = (x_1, \ldots, x_j)$ be the vector of image reservation prices, and let $C_{\mathbf{x}}(s,p_j)$ be the search cost that would leave the searcher indifferent between continuing and stopping search at the point (s,p_j) if the stopping policy from then on is given by the image reservation prices $\mathbf{x}$. Substituting (4-4-3) in (4-4-1), the following expression for $C_{\mathbf{x}}(s,p_j)$ is obtained. The derivation is given in Appendix B.

$$C_{\mathbf{x}}(s,p_j) = \begin{cases} \max\limits_{\mathbf{x}} \dfrac{N(p_j)}{D(p_j)} & j > 1 \\ 0 & j = 1 \end{cases} \tag{4-4-4}$$

where

$$N(p_j) = \left(\frac{s!}{(s-j)!}\right)p_j + \sum_{i=1}^{j-1}\left[\left(\prod_{k=i}^{j-1}\left(x_{k+1}-k\right)\right)\sum_{h=1}^{i}\left(\frac{x_i!\,(x_i-i+1)}{(x_i-i+1)!\,i} - \frac{x_{i+1}!}{(x_{i+1}-i)!\,i}\right)p_h\right], \tag{4-4-5}$$

$$D(p_j) = \frac{(s+1)!}{(s-j)!\,j} + \sum_{i=1}^{j-1}\left(\prod_{k=i}^{j-1}(x_{k+1}-k)\right)\frac{(x_i+1)!\,(x_i-i+1)}{(x_i-i+1)!\,i(i+1)}, \tag{4-4-6}$$

Since consumers choose $\mathbf{x}$ to minimize costs, or, in the dual problem, to maximize $C(s,p_j)$,

$$C(s,p_j) = \max_{\mathbf{x}} C_{\mathbf{x}}(s,p_j)$$

$$\text{subject to } \; x_1=0; \;\; i-1 \le x_i \le x_{i+1} \;\; \forall\; 1<i<j-1; \;\; x_j=s \tag{4-4-7}$$

The restriction that $x_i \ge i-1$ follows from the point (t,p_i) being feasible only for $t \ge i-1$; $x_i \le x_{i+1}$ follows from Proposition 4-2; and $x_j = s$, from $C(s,p_j)$ being defined to give indifference at (s,p_j).

Finally, before using equation (4-4-4) to derive properties of demand functions, a result useful to these derivations is given.

<u>Proposition 4-5:</u>

If $p_j = p_{j+1}$, then $C(s,p_j) = C(s,p_{j+1})$

<u>Proof:</u>

Given in Appendix B.

□

4.5 Demand Functions.

Let consumers have identical inelastic demand for the searched-for good. Their search costs can differ, given by the distribution function H(c). Let $q_j(p_j)$ be the expected demand for firm j.

A firm's price affects its demand in two ways: By influencing the decision of consumers to stop search after sampling other firms, it affects the number of consumers who sample that firm; and it then affects whether those consumers sampling the firm decide to purchase from it or continue sampling.

Now, a consumer who has received a quote lower than p_j will never buy from firm j. Let η_{sj} be the expected number of consumers who sample firm j when there are s other firms remaining to be sampled, and who have not received a quote lower than p_j. These consumers need not be a representative cross-section of all consumers in terms of their search costs, as the probability that a consumer continues search until s firms remain unsampled depends on his search cost. Let H_{sj} be the distribution function of search costs for these consumers. The proportion of the η_{sj} consumers whose search cost is sufficiently low to induce search to continue is therefore $H_{sj}(C(s,p_j))$, while the proportion $\left(1 - H(C(s,p_j))\right)$ will stop and buy from firm j. Therefore,

$$q_j = \sum_{s=0}^{J-1} \eta_{sj}\left(1-H_{sj}(C(s,p_j))\right) \tag{4-5-1}$$

(Recall that J is the total number of firms.)

Now on average, $\frac{1}{J-j}\eta_{sj}$ of the consumers sampling firm j when s firms remain unsampled will have sampled firm i, i>j, when s+1 firms were unsampled. Let H^i_{sj} be the distribution function of search costs for those consumers. Then,

$$H_{sj}(c) = \frac{1}{J-j}\sum_{i=j+1}^{J} H^{i}_{sj}(c) \tag{4-5-2}$$

with
$$H^{i}_{sj}(c) = \left\{ \begin{array}{ll} \dfrac{H_{(s+1)i}(c)}{H_{(s+1)i}(C(s+1,p_i))} & c \le C(s+1,p_i) \\ 1 & c > C(s+1,p_i) \end{array} \right\} \quad s < J+1 \tag{4-5-3}$$

and
$$H_{(J-1)j}(c) = H(c)$$

Likewise, η_{sj} can be found recursively,

$$\eta_{sj} = \frac{1}{s+1}\sum_{i=j+1}^{J} \eta_{(s+1)i} H_{(s+1)i}(C(s+1,p_i)) \tag{4-5-4}$$

with
$$\eta_{(J-1)j} = \frac{M}{J}$$

where M is the total number of consumers.

The demand function for firm j is given by equations (4-5-1) - (4-5-4). The following result follows from these equations.

Theorem 4-1:

If H(c) is continuous and differentiable, then the expected demand for each firm is a continuous monotonic-decreasing function in its own price.

Proof:

Only a price change that leave the ranking of prices unaltered is considered. The continuity implied by Proposition 4-5 ensures that the result applies to global changes as well.

From (4-5-1) - (4-5-4) and the continuity of H, q_j is continuous in each $C(s,p_i)$, $i \ge j$. Continuity of q_j in price then follows from the $C(s,p_i)$ being continuous in price $\forall s$ $\forall$ $i \geq j$, as shown by equation (4-4-4).

Where the derivatives exist,

$$\frac{\partial q_j}{\partial p_j} = \sum_{s=0}^{J-1} \frac{\partial q_j}{\partial C(s,p_j)} \cdot \frac{\partial C(s,p_j)}{\partial p_j} + \sum_{s=0}^{J-1} \sum_{i=j+1}^{J} \frac{\partial q_j}{\partial C(s,p_i)} \cdot \frac{\partial C(s,p_i)}{\partial p_j} \qquad (4\text{-}5\text{-}5)$$

Now, from (4-5-1)

$$\frac{\partial q_j}{\partial C(s,p_j)} < 0 \qquad \forall s; \qquad (4\text{-}5\text{-}6)$$

and from (4-5-3),

$$\frac{\partial H^i_{sj}(c)}{\partial C(s,p_i)} \geq 0 \qquad \forall i>j, \qquad (4\text{-}5\text{-}7)$$

so, from (4-5-1),

$$\frac{\partial q_j}{\partial C(s,p_i)} \geq 0 \qquad \forall i>j, \forall s \qquad (4\text{-}5\text{-}8)$$

Finally, from (4-4-4), where the derivatives exist,

$$\frac{\partial C(s,p_j)}{\partial p_j} > 0 \quad , \quad \frac{\partial C(s,p_i)}{\partial p_j} < 0 \qquad \forall i>j \qquad (4\text{-}5\text{-}9)$$

Substituting (4-5-6), (4-5-8) and (4-5-9) into (4-5-5) shows that $\frac{\partial q_j}{\partial p_j} < 0$.

□

The dual function, C, is defined as the upper envelope of a finite number of functions, C_x. As a result, C is non-differentiable at a finite number of points.

Theorem 4-2:

If the number of firms is finite but exceeds two, the demand functions for each firm will be characterized by both inward and outward kinks—that is, by points where the elasticity of demand for price increases is higher and points where it is lower than for price decreases.

Proof:

From equations (4-4-4) - (4-4-6), note that $C_{\mathbf{x}}(s,p_j)$ is differentiable in p_j. When J=2, the restriction in (4-4-7) implies that **x** takes a single value; C is then equal everywhere to some differentiable function $C_{\mathbf{x}}$. Let $\hat{p}_j$ be a value of p_j where the optimal value of **x** changes for some $C(s,p_i)$, i≥j; that is, a value of p_j where there exists vectors of image reservation prices, **x1** and **x2**, and ε > 0 such that, given all other prices,

$$C(s,p_i) = \begin{cases} C_{\mathbf{x1}}(s,p_j) > C_{\mathbf{x2}}(s,p_j) & p_j = \hat{p}_j + \varepsilon \\ C_{\mathbf{x1}}(s,p_j) = C_{\mathbf{x2}}(s,p_j) & p_j = \hat{p}_j \\ C_{\mathbf{x2}}(s,p_j) > C_{\mathbf{x1}}(s,p_j) & p_j = \hat{p}_j - \varepsilon \end{cases} \qquad (4\text{-}5\text{-}10)$$

When J>2, there will be a finite number of such points $\hat{p}_j$. At any of these, (4-5-10) implies that

$$\lim_{p_j \to \hat{p}_j^+} \frac{\partial C(s,p_i)}{\partial p_j} > \lim_{p_j \to \hat{p}_j^-} \frac{\partial C(s,p_i)}{\partial p_j}$$

At values, $\hat{p}_j$, of p_j where there is non-differentiability in $C(s,p_j)$, then, (4-5-6) gives

$$\lim_{p_j \to \hat{p}_j^+} \frac{\partial q_j}{\partial p_j} < \lim_{p_j \to \hat{p}_j^-} \frac{\partial q_j}{\partial p_j} \qquad \text{an inward (to the origin) kink,}$$

and for values, $\hat{p}_j$, of p_j where there is non-differentiability in $C(s,p_i)$, $i > j$, (4-5-8) gives

$$\lim_{p_j \hat{p}_j^+} \frac{\partial q_j}{\partial p_j} > \lim_{p_j \hat{p}_j^-} \frac{\partial q_j}{\partial p_j} \qquad \text{an outward kink.}$$

□

If a firm has a differentiable cost function, it will never set its price at a point where its demand function has an outward kink. The demand functions produced from this form of search are therefore as troublesome to the existence of equilibria as are the discontinuities resulting from FSS search (which are, in effect, outward kinks) since the best-reply functions of firms will not be continuous. Examples of existent equilibria can be obtained, but any such examples are not robust. For-instance, Stiglitz (1987) has shown that a single price equilibrium is not possible under with-replacement search when $h(0) > 0$ and there are more than two firms. This is because, if all other prices are the same, a firm has an outward kink at that price in its demand function. He does give an example of a dispersed price equilibria with three firms.

4.6 Concluding Remarks.

This chapter has been concerned with demonstrating both the similarities and differences between sequential search models using with-replacement sampling and those using without-replacement. Although some desirable properties of the former model carry over to the latter, namely the reservation-price rule and no-use-of-recall, without-replacement sampling does not lead to nicely behaved demand functions for firms.

It is not the without-replacement assumption *per-se* that leads to demand kinks, but rather the non-stationarity of the problem. non-stationarity requires that later reservation prices be calculated when solving the dual at any point, and so leads to a maximum operator in the definition of C. Stationarity also fails when there are income effects, discounting over time, adaptive learning, or search costs that depend on the number of firms sampled (as, for example, when search costs arise from spatial separation of firms). These cases also lead to kinked demand functions. Equilibrium models which use the standard with-replacement model of sequential search are indeed special.

Chapter 5: Blind Search — Adaptive Sampling With Minimum Information

5.1 Introduction.

A formal specification of the the optimization problem of agents facing an uncertain environment requires that the agents have some information about the uncertainty. For example, under the rational-expectations assumption common to most models of consumer price search, consumers are assumed to know the price distribution; the only uncertainty is over the prices charged by particular firms. This assumption assigns consumers the maximum possible amount of information about prices consistent with a stochastic model. With such a strong assumption, strong results are obtained. The major results of the basic rational-expectations, sequential search model—a stationary reservation price, myopia, and no use of recall—all depend on rational expectations.

In adaptive search models, sequentially-searching consumers have less information. They have prior beliefs about the price distribution, but can update these beliefs as search progresses. Models of adaptive search have been developed by Axell (1974), Rothschild (1974), Kohn and Shavell (1974), Rosenfield and Shapiro (1981), and Morgan (1985).[1] One of the appeals of adaptive models is generality. For instance, the Rothschild, Kohn and Shavell, and Morgan models all include rational expectations as a special case in which the prior belief accords certainty to a particular price distribution. The cost of generality is that adaptive models yield few predictions, for almost any result is possible given appropriate priors. Stronger results can be obtained by placing restrictions on the priors. For instance, Rothschild derives most of the results in his paper for the case where consumers' priors have the Dirichlet distribution.

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[1] Kohn and Shavell consider a general model which includes adaptive search among its possible interpretations.

The adaptive model considered in this chapter, termed "Blind Search" for reference, imposes very strong restrictions on consumers' beliefs about the price distribution. These beliefs represent about the minimum amount of information that can be given to consumers and still leave a problem that can be formalized. The assumption is made that consumers' initial priors are over how prices are related rather than the absolute level of prices—that is, over the second rather than the first moment of the price distribution. This yields strong predictions about searchers' behaviour that, by casual empiricism, seem realistic. Among these is that consumers will be induced to stop search by both wide and narrow ranges of sampled prices. The main motivation for the model, however, is that within the range of search behaviour encompassed by general adaptive models, Rational Expectations Search and Blind Search represent the two extremes in terms of the information available to searchers. The specific results obtainable from the two models can then serve as useful benchmarks for considering the importance of information assumptions in models using search. Blind Search is used as such a benchmark in the following chapter, which considers the robustness of equilibrium results to the rational-expectations assumption.

The assumptions made about consumers' beliefs are motivated in the following section. The model is formally presented in Section 5.3. Section 5.4 establishes properties of Blind Search, which are used to derive comparative static results in Section 5.5, and to make comparisons with Rational Expectations Search in Section 5.6. Some derivations and proofs for this chapter are contained in Appendix C.

5.2 Blind Search.

The reason that the assumption of Rational Expectations seems very strong is not that it is unrealistic for consumers to know the price distribution completely, but that they cannot have this information without also knowing the prices charged by specific firms. Until they acquire such specific information by search, consumers can only have vague prior beliefs about the price distribution.

The prices observed by search do, however, give information about the distribution of the unsampled prices. It is reasonable for searchers to assume that the prices charged by different firms for a homogeneous good have positive correlation; that is, the observation of a low price makes it more likely that other prices are low. If prices are correlated, then those known as a result of the search process act as a signal about the prices charged by the remaining firms. One way of modeling this explicitly is to assume that all prices are perceived by searchers to be random variables independently drawn from the same distribution. As prices are sampled, information is gathered about the distribution from which they are assumed to be drawn.[2]

The interpretation of prices as signals suggests a way of modeling searchers' prior beliefs. Consumers only have a rough idea of the absolute level of prices, but past experience in other markets gives them some idea on how the prices in a particular market are likely to be related. It seems natural, then, to assume that consumers' priors are not over absolute prices, but over higher moments of the distribution. The assumption, that priors are over higher moments, implies that no beliefs about the price distribution itself are required to formulate the optimization problem. This and the fact

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[2] In most search models, the population about which searchers' beliefs are formed is the actual distribution of prices charged by firms. Here, uncertainty is captured by assuming actual prices to be random variables drawn from an unobservable population. When there is an infinity of firms, there is no difference between the approaches. When there is a finite number of firms, the present approach allows with-replacement statistical theory to be applied to without-replacement behaviour. It also allows the price distribution to be considered continuous without compromising internal consistency when considering a small number of firms.

that uniform distributions are used for the priors that are required are the features that make Blind Search a minimum information model. These features are presented in detail in the next section.

As usual, although the language of buyers searching for the lowest price is used, variables can be reinterpreted to apply to seller search, searching for utility etc. The only qualification is that the search cost c is expressed in the same units as the searched for data (e.g. dollars, utils ...) and must be constant over search. This constraint in effect assumes away discounting and income effects.

5.3 Notation and Set-up of the Model.

A. The Sampling Environment.

There are J firms selling a homogeneous good. The consumer is assumed to sample these firms sequentially, without replacement to find the lowest price. He wants to buy one unit of the good. The consumer always samples at least one firm. After each quotation he must decide whether to search an additional firm at cost c, or to purchase at the lowest quotation received up to that stage; that is, full recall is assumed. His criterion for this decision is to minimize expected expenditure. Because search is without replacement, J is the maximum number of quotations that can be sought.

B. The Consumer's Prior.

The consumer regards all prices as being i.i.d. random variables, each drawn from the same distribution with unknown range $\hat{x}$. Buyers have a prior belief that $\hat{x}$ is itself uniformly distributed, over the range [0,X]. X is known; it is the only information about the price distribution that consumers have prior to search.

There are two sources of uncertainty here: the uncertainty arising because prices are random variables, and the uncertainty over $\hat{x}$. The first is common to all search models. If the magnitude of $\hat{x}$ and its location on the real line were known (that is, the price support), then this structure would be formally equivalent to rational-expectations search from a uniform continuum of prices where the searcher was limited to taking no more than J quotations. It is the uncertainty over $\hat{x}$ that makes this model adaptive. The quotations received enable the prior on $\hat{x}$ to be updated and also give information on the absolute level of prices.

Note the sense in which this is a minimum information model. *All* the information about prices comes from sampling, since these assumptions imply that the original prior on *prices* is an improper uniform distribution over the range $[-\infty, \infty]$. The prior about the distribution of the price range allows the posterior price distribution to be a proper distribution over a bounded support. For instance, after one price, p_1, has been sampled, the posterior price distribution is defined over $[p_1-X, p_1+X]$.

The assumptions made about consumers' beliefs imply that, before search, consumers think positive and negative prices to be equally likely. Although unrealistic, this keeps the analysis simple. An additional constraint that the searcher expects all prices to be non-negative would only affect the results when sampled prices were low. As long as the maximum price sampled exceeds X, the support of the posterior distribution is positive everywhere, so an assumption of non-negative prices would be redundant.

C. The Consumer's Problem.

Let $\{p\}^n$ be the set of prices, $\{p_1, p_2, \ldots p_n\}$, observed after n firms have been sampled, let $p_{n\ell}$ and p_{nh} be the minimum and maximum prices in this sequence, and let x_n be the sample range

$$x_n \equiv p_{nh} - p_{n\ell}.$$

$F_n(p|\{p\}^n)$ is the distribution function of the (n+1)th price after the information in $\{p\}^n$ is used to update prior beliefs.

The consumer's problem is to find the stopping rule that minimizes expected expenditure. Let $A_n(\{p\}^n)$ be the expected cost (from additional search and purchase) for a consumer who has observed $\{p\}^n$, assuming the optimal stopping policy is followed at that and every succeeding point. Then

$$A_n(\{p\}^n) \equiv \min\left\{ p_{n\ell},\ \left(\int A_{n+1}(\{\{p\}^n, p\})\cdot dF_n(p|\{p\}^n) + c\right)\right\} \qquad (5\text{-}3\text{-}1)$$

With the usual convention that a consumer indifferent between stopping and continuing search will stop, the optimal stopping rule can be expressed: *Stop search if and only if* $A_n(\{p\}^n) = p_{n\ell}$.

This, of course, implies continuing search if and only if the marginal benefit exceeds the marginal cost c. Since c is constant, it will be convenient to characterize the consumer's problem at every point by the marginal benefit of taking an additional quotation. Let $G_n^*(x_n)$ be the marginal benefit after n quotations, gross of the search cost, of taking an (n+1)th quotation assuming the optimal policy is followed from then on. Then,

$$G_n^*(x_n) \equiv p_{n\ell} - \int A_{n+1}(\{\{p\}^n, p\})\cdot dF_n(p|\{p\}^n), \qquad (5\text{-}3\text{-}2)$$

and the optimal stopping rule can be reexpressed in terms of $G_n^*(x_n)$: *Stop search if and only if* $G_n^*(x_n) \le c$. G_n^* is written as a function of the sample range, x_n, rather than $\{p\}^n$ since, as will be shown in the next section, the particular prior beliefs assumed here imply that x_n is a sufficient statistic to determine the stopping rule. That is, $\{p\}^n$ only affects the stopping rule through its effect on x_n.

5.4 Properties of Blind Search.

The properties of $G_n^*(x_n)$ are derived in three stages as follows:

a) Let $\hat{G}_n(x_n)$ be the marginal benefit when $\hat{x}$ is known for a searcher taking one and only one additional quotation. Then

$$\hat{G}_n(x_n) = p_{n\ell} - \int^{p_{n\ell}} p \cdot d\hat{F}_n(p|\{p\}^n) - \int_{p_{n\ell}} p_{n\ell} \cdot d\hat{F}_n(p|\{p\}^n) \qquad (5\text{-}4\text{-}1)$$

where $\hat{F}_n(p|\{p\}^n)$ is the distribution function of p_{n+1} when $\hat{x}$ is known. Because knowing $\hat{x}$ is equivalent to knowing the distribution of a uniform continuum of prices, this is the standard rational-expectations, sequential search problem.[3]

b) Let $G_n(x_n)$ be the marginal benefit of a single quotation when $\hat{x}$ is unknown, so

$$G_n(x_n) = p_{n\ell} - \int^{p_{n\ell}} p \cdot dF_n(p|\{p\}^n) - \int_{p_{n\ell}} p_{n\ell} \cdot dF_n(p|\{p\}^n)$$

$$= p_{n\ell} - \int p_{(n+1)\ell} \cdot dF_n(p|\{p\}^n). \qquad (5\text{-}4\text{-}2)$$

Let $p(\hat{x}|\{p\}^n)$ be the posterior density function of $\hat{x}$ when the searcher has information $\{p\}^n$. Then,

$$F_n(p|\{p\}^n) = \int \hat{F}_n(p|\{p\}^n) \cdot p(\hat{x}|\{p\}^n)\, d\hat{x},$$

so from (5-4-1) and (5-4-2)

---

[3] This is true because the standard problem is *myopic*; that is, the searcher's stopping rule can be found by considering the expected value of taking one more quotation.

$$G_n(x_n) = \int_{x_n}^{X} \hat{G}_n(x_n)\cdot p(\hat{x}|\{p\}^n)\ d\hat{x}. \tag{5-4-3}$$

c) The true value of additional search $G_n^*(x_n)$ differs from $G_n(x_n)$ as it takes into account the information an additional quotation will contain about the value of searching yet further when searchers are not constrained to take only one more quotation. From (5-3-2) and (5-3-1)

$$G_n^*(x_n) = p_{n\ell} - \int\left[\min\left\{ p,\ p_{n\ell},\ \left(\int A_{n+2}(\{\{p\}^n,p,p'\})\cdot dF_{n+1}(p'|\{\{p\}^n,p\}) + c\right)\right\}\right]\cdot dF_n(p|\{p\}^n)$$

$$= p_{n\ell} - \int\left[p_{(n+1)\ell} + \min\left\{ 0,\ \left(\int A_{n+2}(\{\{p\}^n,p,p'\})\cdot dF_{n+1}(p'|\{\{p\}^n,p\}) + c\right) - p_{n\ell}\right\}\right]\cdot dF_n(p|\{p\}^n)$$

$$= G_n(x_n) + \int \max\left\{ 0,\ p_{n\ell} - \left(\int A_{n+2}(\{\{p\}^n,p,p'\})\cdot dF_{n+1}(p'|\{\{p\}^n,p\}) + c\right)\right\}\cdot dF_n(p|\{p\}^n)$$

(from (5-4-2))

$$= G_n(x_n) + \int \max\left\{ 0,\ \left(G_{n+1}^*(x_{n+1}) - c\right) \right\}\cdot dF_n(p|\{p\}^n) \tag{5-4-4}$$

With $G_n^*(x_n)$ expressed as a dynamic program derived from $G_n(x_n)$, properties of $G_n(x_n)$ can be extended to $G_n^*(x_n)$ by backward induction. These three stages are now developed fully.

A. $\hat{G}_n(x_n)$.

Because there is full recall, the value of continuing search depends only on the distribution of prices below the lowest price already sampled, $p_{n\ell}$. It is well known from the search literature that[4]

$$\hat{G}_n(x_n) = \int^{p_{n\ell}} \hat{F}_n(p|\{p\}^n)\ dp \qquad (5\text{-}4\text{-}5)$$

If $\hat{x}$ is known, prices are uniform i.i.d. random variables distributed over some interval $[a, a+\hat{x}]$, giving a distribution of prices, conditional on a,

$$\frac{p-a}{\hat{x}} \qquad a \le p \le a+\hat{x}. \qquad (5\text{-}4\text{-}6)$$

Now a is itself uniform, over $[p_{n\ell}-(\hat{x}-x_n),\ p_{n\ell}]$. (The limits arise because the population minimum, a, cannot exceed the sample minimum, $p_{n\ell}$; and the sample maximum, $p_{n\ell}+x_n$, cannot exceed the population maximum, $a+\hat{x}$.) Therefore,

$$\hat{F}_n(p|\{p\}^n) = \int_{p_{n\ell}+x_n-\hat{x}}^{p} \frac{(p-a)}{\hat{x}}\cdot\frac{1}{(\hat{x}-x_n)}\ da \qquad \text{for } p_{n\ell}+x_n-\hat{x} \le p \le p_{n\ell}$$

(The upper limit of the integration arises from the restriction in (5-4-6) that $p \ge a$.) This integral gives

$$\hat{F}_n(p|\{p\}^n) = \frac{(p-p_{n\ell}+(\hat{x}-x_n))^2}{2\hat{x}(\hat{x}-x_n)} \qquad \text{for } p_{n\ell}+x_n-\hat{x} \le p \le p_{n\ell} \qquad (5\text{-}4\text{-}7)$$

Equation (5-4-7) into (5-4-5) gives

$$\hat{G}_n(x_n) = \int_{p_{n\ell}+x_n-\hat{x}}^{p_{n\ell}} \frac{(p-p_{n\ell}+(\hat{x}-x_n))^2}{2\hat{x}(\hat{x}-x_n)}\ dp$$

---

[4] For example see Rothschild (1974); the formula derives from equation (5-4-1), integrating by parts.

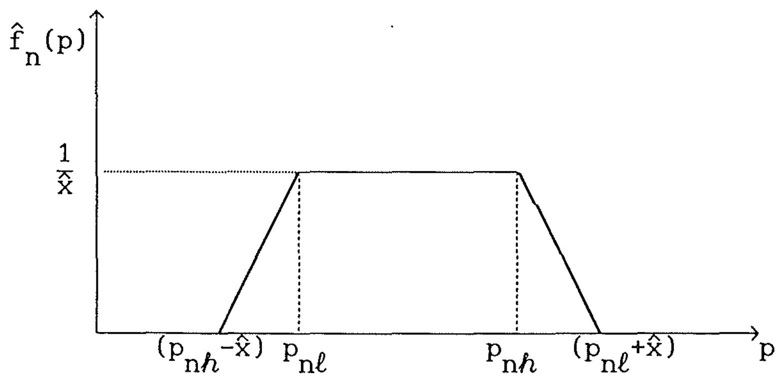
$$= \frac{(\hat{x}-x_n)^2}{6\hat{x}} \tag{5-4-8}$$

For later reference, the full p.d.f. of prices conditional on $\hat{x}$, $\hat{f}_n(p|\{p\}^n)$, is given here.

$$\hat{f}_n(p|\{p\}^n) = \begin{cases} \dfrac{p-p_{n\ell}+\hat{x}-x_n}{\hat{x}(\hat{x}-x_n)} & p < p_{n\ell} \\ \dfrac{1}{\hat{x}} & p_{n\ell} \le p \le p_{nh} \\ \dfrac{\hat{x}-x_n+p_{nh}-p}{\hat{x}(\hat{x}-x_n)} & p_{nh} < p \end{cases} \tag{5-4-9}$$

This is illustrated in Figure 5-1. For $p<p_{n\ell}$, $\hat{f}_n(p)$ is derived from (5-4-7). From the set-up of the problem, prices above p_{nh} are as likely as those below $p_{n\ell}$, so the density is symmetric. Between $p_{n\ell}$ and p_{nh}, the distribution of observed prices contains no information; all prices within this range are equally likely to be the next price observed.

Figure 5-1.



B. $G_n(x_n)$.

Let $\hat{p}(x_n|\hat{x})$ be the conditional density function of x_n. It is shown in Appendix C that

$$\hat{p}(x_n|\hat{x}) = n(n-1)\frac{x_n^{n-2}}{\hat{x}^{n-1}}\left(1-\frac{x_n}{\hat{x}}\right)$$

By Bayes' Theorem,

$$p(\hat{x}|x_n) = \frac{p(x_n|\hat{x})\cdot p(\hat{x})}{\int_{x_n}^{X} p(x_n|x')\cdot p(x')dx'}$$

$$= \frac{\frac{1}{\hat{x}}\left(\frac{x_n}{\hat{x}}\right)^{n-2}\left(1-\frac{x_n}{\hat{x}}\right)}{\int_{x_n}^{X} \frac{1}{x'}\left(\frac{x_n}{x'}\right)^{n-2}\left(1-\frac{x_n}{x'}\right)dx'} \tag{5-4-10}$$

(since $p(\hat{x}) = p(x') = \frac{1}{X}$).

The following substitutions simplify the integrals:

Let $R = \left(\frac{x_n}{X}\right)$, $\hat{R} = \left(\frac{x_n}{\hat{x}}\right)$, and $R' = \left(\frac{x_n}{x'}\right)$. Then,

$$p(\hat{x}|x_n) = \frac{\frac{1}{\hat{x}}\left(\hat{R}^{n-2}\right)\left(1-\hat{R}\right)}{\int_{R}^{1} R'^{\,n-3}(1-R')dR'} \tag{5-4-11}$$

When $\hat{x}$ and x_n are known, all distribution of n prices giving that x_n are equally likely. It follows that x_n exhausts the information about $\hat{x}$ contained in $\{p\}^n$, and hence that x_n is a sufficient statistic to update the distribution of $\hat{x}$.

Substituting $\hat{R}$ for $\frac{x_n}{\hat{x}}$ in equation (5-4-8), and the result and (5-4-11) into (5-4-3), gives

$$G_n(x_n) = \frac{XR\int_{R}^{1} \hat{R}^{n-4}(1-\hat{R})^3 d\hat{R}}{6\int_{R}^{1} R'^{\,n-3}(1-R')dR'} \tag{5-4-12}$$

We continue to use the substitute variables R, R′ and $\hat{R}$ purely for typographical convenience. In this form it is clear that $G_n(x_n)$ is homogeneous of degree 1 in all prices, since R is independent of the units of measurement.

$G_n(x_n)$ depends only on the observed price range, and not the values of specific prices. This arises because the sample range is a sufficient statistic to update prior beliefs, and because the consumer, in deciding whether to continue search, is interested in the expected *reduction* in the minimum price rather than the minimum price itself. This feature is particularly important when n=1. $G_1(x_1)$ only has an interpretation at $x_1=0$, since the sample range must always equal zero after one observation. Solving equation (5-4-12) for n=1 gives

$$G_1(x_1) = \frac{X(1-6R+3R^2+2R^3-6R^2\ln R)}{12(1-R+R\ln R)}$$

with

$$\lim_{x_1 \to 0} G_1(x_1) = \frac{X}{12}. \qquad (5\text{-}4\text{-}13)$$

Because the searcher has no prior beliefs on the absolute level of prices, the first quotation gives no information on the spread of prices. Put another way, the price sample loses a degree of freedom in estimating the population mean. As a result, the decision to seek a second quotation is independent of the first.

Blind Search shares with Rational Expectations Search the simplicity of depending on a single sample statistic—in this case the sample range, rather than the sample minimum. The optimal search strategy of consumers can then be easily described. A number of results are intuitive: When $x_n=X$ ($\Rightarrow$ R=1), the maximum price range the consumer believes is possible has been observed. As the minimum price must then have been sampled, there can be no gain from further search. If $x_n=0$, for $n\geq 2$, there is zero probability that $\hat{x}$ exceeds zero; again price reduction from further search is impossible. As the number of firms sampled increases, so does the probability that the sample range x_n is close to the true value $\hat{x}$, and hence the gain from further search decreases. These intuitions are confirmed by the following proposition.

<u>Proposition 5-1:</u>

For $n \geq 2$, and for all $x_n \in (0,X)$:

a) $G_n(0) = \lim_{x_n X} G_n(x_n) = 0$;

b) G_n is strictly quasi-concave in x_n;

c) G_n is "star-shaped" at the origin — i.e. $\dfrac{\partial G_n(x_n)}{\partial x_n} < \dfrac{G_n(x_n)}{x_n} \quad \forall\ x_n$;[5]

d) $G_n(x_n) > G_{n+1}(x_n)$.

<u>Proof:</u>

Given in Appendix C.

□

The quasi-concavity of $G_n(x_n)$ and its zero value at the endpoints provides a useful counterpart in Blind Search to the reservation-price rule of Rational Expectations Search.

<u>Proposition 5-2:</u>

a) For each n, there exists *reservation range* values, $x_{n\ell}$, x_{nh}, $0\leq x_{n\ell}\leq x_{nh}\leq X$, such that $G_n(x_n)>c$ if and only if $x_n\in(x_{n\ell},x_{nh})$.

b) The reservation range values converge as n increases in the sense that if $x_{n\ell} < x_{nh}$, then $x_{(n-1)\ell} < x_{n\ell}$ and $x_{(n-1)h} > x_{nh}$.

---

[5] A function f is star-shaped at a point x if the line from f(x) to any other point on the function lies below the function.

Proof:

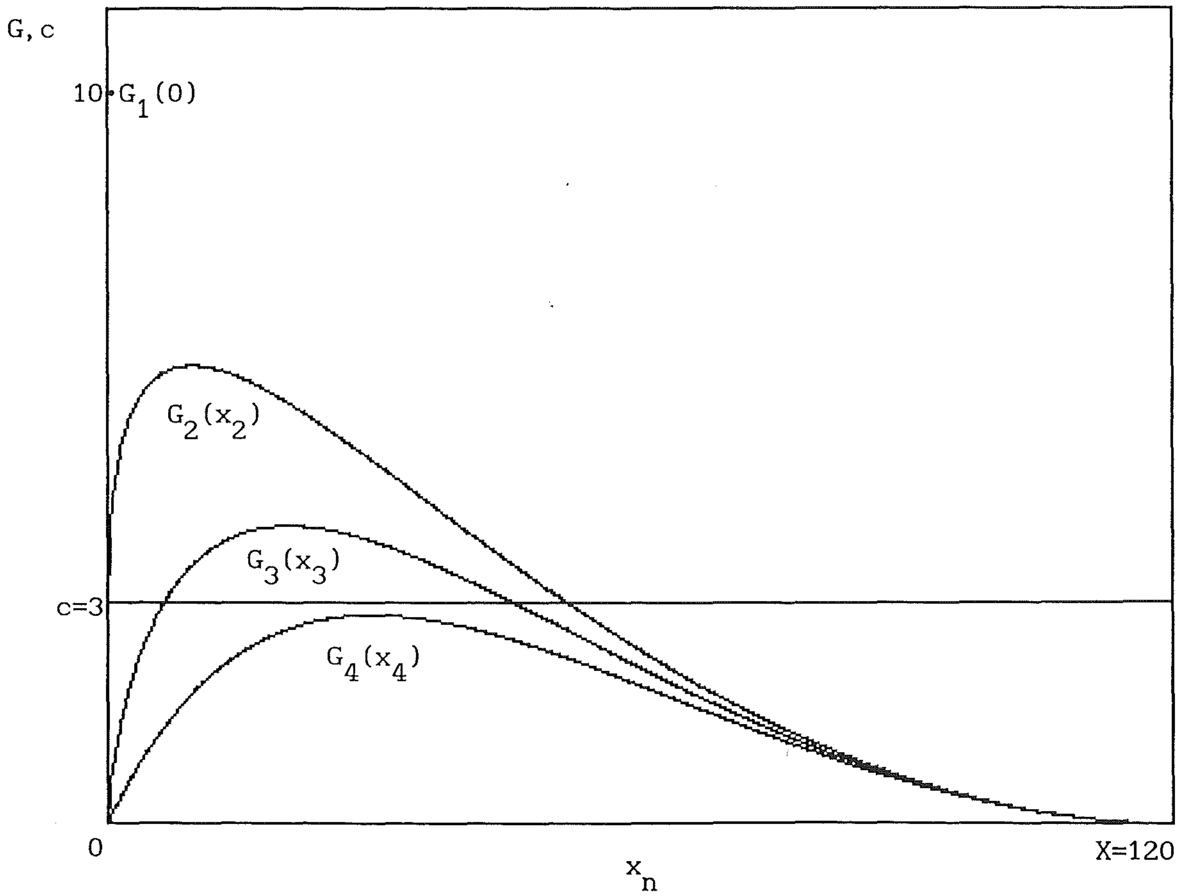
Follows from a), b) and d) of Proposition 5-1.

□

Note that $x_{n\ell}$ and x_{nh} are clearly unique if $x_{n\ell} \neq x_{nh}$. However, if $\max\{G_n(x_n)\} < c$, then any value of $x_{n\ell}$ will satisfy the above definition as long as $x_{n\ell}=x_{nh}$.

Propositions 5-1 and 5-2 are illustrated in Figure 5-2 for X=120 and c=3. From (5-4-13), these particular values give $G_1(0) = 10 > c$, so the searcher would always take at least two quotations. $x_{2\ell}$, $x_{3\ell}$, and x_{2h}, x_{3h} are found where $G_2(x_2)$ and $G_3(x_3)$ cross the horizontal line c=3 from below and above respectively. Values for these are given below the figure.

Figure 5-2.



$x_{2\ell} \cong 0.60$ $x_{3\ell} \cong 6.48$ $x_{3h} \cong 45.60$ $x_{2h} \cong 51.84$

C. $G_n^*(x_n)$.

To complete the description of consumer's search behaviour, it remains to show that the reservation range property holds when consumers are not constrained to seek only one more quotation.

Let $p_n(x|x_n)$ be the mixed probability distribution and density function of x_{n+1} given x_n; and let $\mathbb{X}_n$ be the set

$$\mathbb{X}_n = \left\{ x \,|\, (G^*_{n+1}(x)-c)>0;\ 0\le x\le X \right\}$$

Now that x_n has been shown to be a sufficient statistic to calculate the stopping rule, equation (5-4-4) can be rewritten

$$G^*_n(x_n) = G_n(x_n) + \int_{x\in\mathbb{X}_n} \left(G^*_{n+1}(x)-c\right)\cdot p(x|x_n)dx. \tag{5-4-14}$$

Since $p(x|x_n)$ is a mixed distribution, the integral in equation (5-4-14) is interpreted as requiring both summation and integration.

Using equation (5-4-14), the results of Propositions 5-1 and 5-2, can be extended to search not constrained to a single additional quotation.

<u>Proposition 5-3:</u>

a) There exist reservation range values $x^*_{n\ell}$, x^*_{nh}, $0\le x^*_{n\ell}\le x^*_{nh}\le X$, such that search will continue if and only if $x_n \in (x^*_{n\ell}, x^*_{nh})$.

b) Search intensity falls as n increases in the sense that if $x^*_{n\ell} < x^*_{nh}$, then $x^*_{(n-1)h} > x^*_{nh}$, and that $x^*_{n\ell}$ converges (though not necessarily monotonically) to x_{nh} as n increases.

<u>Proof:</u>

Given in Appendix C.

□

5.5 Comparative Statics.

A feature of the basic rational-expectations, sequential search model is that a consumer will continue sampling until a price not exceeding the reservation price is found. In Blind Search, no matter how many firms there are, the number sampled is bounded.

Theorem 5-1:

If the search cost is positive, there is a finite number n^*, such that no more than n^* firms will be sampled, whatever the sequence of prices sampled.

Proof:

From (5-4-12),

$$\lim_{n\,\infty} G_n(x_n) = 0 \qquad \forall\, x_n \in [0,X].$$

Therefore, if $c > 0$, there exists n^* such that

$$G_n(x_n) \le c \qquad \forall\, x_n \in [0,X],\ \forall\, n > n^*;$$

and hence from (5-15),

$$G_n^*(x_n) \le c \qquad \forall\, x_n \in [0,X], \quad \forall\, n > n^*;$$

□

In the example of Figure 5-2, $n^* = 3$.

For $n < n^*$, the reservation-range bounds provide a simple measure of search intensity, suitable for comparative static analysis: A diverging of the reservation range values $x^*_{n\ell}$, x^*_{nh} implies a greater intensity of search.

There are three parameters that a consumer searching blindly considers in determining whether to stop search: his search cost, the total number of firms, and the perceived maximum price range. The following theorem describes the effect of changes in the other three parameters.

<u>Theorem 5-2:</u>

Search intensity increases as:

a) the unit search cost decreases;

b) the total number of firms increases, up to some critical number J^*;

c) the perceived variability of price increases.

<u>Proof:</u>

The formal proof is left for Appendix C, but the intuition is outlined here.

The effect of decreasing the search cost is obvious. The consumer searches if and only if the benefit from search exceeds the cost. Lowering the cost increases the range of x_n for which that will apply.

The difference between $G_n^*(x_n)$ and $G_n(x_n)$ represents the value of the information about the price distribution that is contained in a price quotation. The more firms that remain to be sampled, the greater is the potential value of this information. The limit, J^*, is equal to n^* of Theorem 5-1. Since no more than n^* firms will ever be sampled, and since n^* is independent of J, increasing firm numbers beyond this point will be of no benefit to the consumer.

Because of the homogeneity of degree 1 of $G_n(x_n)$ (and hence $G_n^*(x_n)$), it is obvious that an increase in X will have the same effect on the normalized bounds, $x^*_{n\ell}/X$, x^*_{nh}/X as a decrease in c. $\partial x^*_{n\ell}/\partial X > 0$ then follows. The expected but less obvious result $\partial x^*_{nh}/\partial X < 0$ can also be shown.

□

5.6 Comparisons with Rational-Expectations Search.

Rothschild (1974) lists four major results of rational-expectations sequential search which carry over to adaptive search when consumers' priors take the Dirichlet distribution:

a) The probability of a searching customer purchasing from a firm is a non-increasing function of the price it charges;

b) As search costs increase, the amount of search decreases;

c) A mean preserving increase in the dispersion of prices reduces total expected costs to the consumer (total search cost plus price);

d) A mean preserving increase in the dispersion of prices increases the amount of search.[6]

In Rational Expectations Search, the *perceived* and *actual* dispersions of prices are the same. In adaptive search models, the perceived price distribution gives the stopping *rule* (that is, the function of observes prices that determines when to stop); the actual distribution affects search by affecting which prices are observed. In Blind Search, for example, the perceived distribution is given by X, the actual distribution affects x_n. Theorem 5-2 c) shows that for an increase in the perceived dispersion of prices, the third and fourth properties of Rothschild hold for Blind Search. Theorem 5-2 a) shows that the second property holds. For the first property, and for the third and fourth when referring to an increase in the actual price dispersion, it is easy to construct counterexamples.

<u>Theorem 5-3:</u>

If all consumers use Blind Search, a firm may face an increasing demand function.

<u>Proof:</u>

Consider a market with three firms 1,2,3, charging prices $p_1 < p_2 < p_3$. Neither firm 1 nor firm 3 can affect their demand with local price changes.

---

[6] Properties a)-c) also apply to rational-expectations, FSS search.

Firm 2, however, by narrowing the gap p_3-p_2, can induce more of the consumers who search firms 2 and 3 first to stop sample them, increasing firm 2's demand at the expense of firm 1.

□

Theorem 5-4:

A mean-preserving increase in the dispersion of actual prices may raise total expected costs to the consumer.

Proof:

Let there be three firms, and consider two distributions:

i) p_1=2, p_2=9, p_3=10; and
ii) p_1=1, p_2=10, p_3=10.

Distribution ii) has the same mean but is more dispersed than distribution i). Now let c be sufficiently small so that $G_2^*(1) > c > 0$. Then for the the first distribution, search will always continue until p_1 is found giving E[p]=2, but for the second, there is a probability of 1/3 that search will terminate after firms 2 and 3 have been searched, giving E[p] = 4. There is a slightly smaller expected search cost in the second case, but this is dwarfed by the difference in expected price.

□

Theorem 5-5:

An increase in actual price-dispersion may decrease the intensity of search.

Proof:

Consider the case where all firms charge either p_1 or p_2, and p_2-p_1=X; compared to one where prices are spread evenly between these two extremes. In the first case, no consumer will search more than twice, while in the second, they may do so if the search cost is sufficiently small.

□

It is known from the Rothschild paper that such counterexamples can be created when priors are not Dirichlet. Although they show that some of the strong results of the standard model are not robust to the removal of the rational-expectations assumption, these counterexamples do use price distributions that are extreme, since their essence is that the observed prices send a bad signal as to the true distribution. Also, such counterexamples would be non-observable, as those price distributions would not occur in an equilibrium. The counterexamples therefore do not seem to represent a major difference between Rational Expectations Search and Blind Search.

Blind Search does, however, yield three definite predictions that are not features of Rational Expectations Search, but that seem realistic by casual empiricism. These are that both low and high ranges of sampled prices will induce search to stop, that the decision to sample a second firm is independent of the price quoted by the first, and that there is an upper limit to the number of firms that will be sampled.

That search will stop for low sample ranges reflects the fact that there is sometimes not enough price dispersion to warrant search but that the consumer only finds this out by searching. For example, if a searcher receives a number of identical quotes, he may conclude that there is retail price maintenance and stop searching. The prediction of the model that it only requires two identical prices to terminate search seems overly strong. This result follows not from the basic formulation of consumer behaviour, but rather from the assumption that the consumer perceives prices as having been drawn from a continuous distribution. Of course within any range, there are only a finite number of possible prices (multiples of one cent), and even fewer "focal-point" prices (for example, the manufacturer's suggested price, multiples of $100 -1, etc.). With a discrete perceived price distribution, the marginal benefit of search at $x_n=0$ would exceed zero, so search would not necessarily terminate.

The independence of the decision to seek a second quotation is a consequence of the consumer not having prior beliefs about absolute prices. A similar result is found in Axell's (1974) adaptive search model where searchers have prior beliefs on the variance of prices but not the mean. Jan Herin, the discussant for this paper, suggested that Axell's assumption of the absence of a subjective conception of the mean is unrealistic. The interpretation of this assumption, however, is not that searchers have no prior beliefs about absolute prices. It is rather that those beliefs are sufficiently flimsy that the first quotation will be used correct beliefs rather than as a guide to where that price ranks in the distribution.

The limit to the number of firms that will be sampled is a result of search being adaptive. In Rational Expectations Search, a searcher who has received a long sequence of prices higher than the reservation price will not revise his beliefs about the distribution, but will regard himself as unlucky and search on. It can be a very costly policy for a consumer who does not know the distribution to search as if he did, particularly if the consumer's perceived distribution is more heavily weighted to lower prices than is the actual distribution. Gastwirth (1976) analysed the case where searchers guess the price distribution incorrectly. He showed that FSS strategies are considerably more robust than sequential strategies in the sense of having a smaller loss from assuming the wrong distribution. He then suggested a strategy that would reduce this loss: the *Naive Truncated Rule*. With this strategy, consumers predetermine a maximum number of firms to sample and then search sequentially with a reservation price rule up to that number. For this to be an appropriate strategy, consumers would have to believe that more than one price distribution were possible. Adaptive models make this belief explicit in their use of prior distributions. The limit to the number of firms sampled, imposed by Gastwirth, emerges as a result in Blind Search.

This last point indicates a similarity between Blind Search and FSS search. In fact, in some ways, Blind Search is more similar to FSS search than to a simple sequential model where searchers sample as if they know the price distribution. This claim is expanded on in the next chapter which considers non-rational-expectations search in equilibrium.

Chapter 6: Equilibrium Search Without Rational Expectations

6.1 Introduction.

The behaviour of searchers who learn about the price distribution while searching has been comprehensively described by the adaptive search literature dating back to to 1974 with the models of Rothschild, Kohn and Shavell, and Axell. The model of Blind Search in Chapter 5 adds to this literature. Equilibrium-search models, however, have almost exclusively assumed that consumers have rational expectations; that is, that they know the distribution of prices before beginning to search.[1] This assumption makes sense if search is used to compare results between oligopoly models with perfect information and those without. As suggested in the Chapter 5, to assume rational expectations about the price distribution is to make the minimum deviation possible from the assumption of perfect information about prices. The new results that arise from introducing incomplete price information can then be clearly attributed to this change in assumption.

The assumption that consumers can know a price distribution without knowing the prices charged by specific firms is, however, clearly unrealistic. It is reasonable to ask whether the results of equilibrium models apply with weaker assumptions about the way consumers become informed.

The question of the sensitivity of results to the rational-expectations assumption has mainly been addressed in the pure-search literature. For instance, the adaptive models of Rothschild (1974) and Kohn and Shavell (1981) were largely concerned with making comparisons with rational-expectations models. The specific differences between rational expectations and adaptive search enumerated in Chapter 5 are in contrast to the conclusions of these papers. As remarked there, Rothschild showed that four major results of rational-expectations search still hold in his adaptive model when searchers

[1] Exceptions are Wilde and Schwartz (1979), and Gabszewicz and Garella (1985).

have Dirichlet priors over the price distribution. Similarly, Rosenfield and Shapiro (1981) present a broad range of examples where properties such as the reservation-price and myopia still hold with adaptive search. Results such as these enable McKenna to claim:

> *"By and large the additional complexity of adaptive models has not been justified by the additional insights they offer."*
>
> McKenna (1987a, p 104)

Rothschild makes a stronger claim:

> *"...economists can without great loss assume that the qualitative properties of demand functions which arise from optimal search from unknown distributions are the same as those which arise from optimal search from known distributions."*
>
> Rothschild (1974, p 710)

Since these claims are based on the results of pure-search models, they are only applicable to descriptions of consumers' reactions to exogenous price distributions. If search is used in an equilibrium model, then the interdependence between searchers' and firms' actions has to be considered. Rational expectations is not just a special case of adaptive search where the consumer's belief in a particular distribution is unshakable: it also assumes that the belief is *correct*. As a result, if a firm changes its price, consumers are assumed to realize this and adjust their search strategies accordingly. The "qualitative properties of demand functions" can be very different when consumers' search behaviour is thus affected compared to when it is not. Consequently equilibria can be very different.

The effect on equilibrium of not assuming rational expectations is considered generally in Section 6.2, and then illustrated in Section 6.3 using the example of Blind Search. It is shown in these two sections that many of the results of equilibrium-search models depend on the assumption of rational expectations. This raises the question, What is the alternative to this assumption? McKenna's point that adaptive models involve additional complexity is particularly pertinent if search is used to derive firms' demand functions. Even in Blind Search, where very strong assumptions are made about prior beliefs, the resulting description of searchers' behaviour is too unwieldy to make it feasible to derive demand functions. It would be useful if there were a simple non-adaptive search strategy that was both manageable and preserved the major equilibrium properties of adaptive search. In Section 6.4, Blind Search is used to show that a form of FSS sampling may be the non-adaptive model that best meets this objective. The concluding remarks of Section 6.5 suggest directions for further research that could overcome some of the difficulties suggested here.

6.2 Rational Expectations and Equilibrium.

In this section and in Section 6.3, rather than developing a particular equilibrium model, we consider some general properties of non-rational-expectations search if it used in an equilibrium-search model belonging to the general class outlined in Section 2.1, p5. The characteristics of this class are restated here:

Consumers search in a market for a homogeneous good. They may differ in characteristics such as search cost. The firms selling the good follow Bertrand/Nash behaviour with respect to each other, choosing their price to maximize the profits from expected demand, while taking the other prices as given; and follow Stackelberg behaviour with respect to consumers, taking their search strategy as given. Firms may differ in their cost functions, and their number may be fixed or free entry assumed. Firms are labeled according to price so that if there are J firms,

$$p_1 \le p_2 \le \cdots \le p_J.$$

Consumers' demand for the good can be elastic or inelastic in the purchase price. Even if *consumer* demand is inelastic, firms may face a downward sloping demand curve for their own output if price increases induce fewer consumers to buy from them.

The important properties of non-rational-expectations search when considering equilibrium are contained in the following two remarks.

Remark 6-1:

If consumers do not have rational expectations, a firm cannot, by changing its price, affect the number of consumers visiting it.

When searchers have rational expectations, if a firm changes its price, consumers *yet to visit that firm* know that the price distribution is changed and so may change their reservation price or, in a FSS model, their sample size. With non-rational-expectations search, price changes can only affect whether a consumer who visits a firm will decide to buy from it.

It is because this difference between models with and without rational-expectations is not captured in pure search models that they cannnot be used to make inferences about the sensitivity of equilibria to the rational-expectations assumption. Many equilibrium results depend entirely on the ability of firms to affect consumers' reservation prices or the size of the sample they take. For example, Stiglitz (1987) shows that increasing the number of firms can reduce competition (raise prices) in a market. The intuition is that the number of firms affects the significance of the change to the price distribution when a single firm changes its price. This is important because part of the incentive for a firm to lower its price below its competitors is that this action lowers consumers' reservation prices and so induces more consumers to visit it; the more firms there are, the less effect has any one firm on reservation prices and so the less elastic is its demand curve. Similar results in Chapter 7 also depend on price changes affecting the reservation prices of consumers.

The second equilibrium property of non-rational-expectations search follows from Remark 6-1.

<u>Remark 6-2:</u>

If consumers have full recall, the number of buyers buying from the lowest priced firm is locally inelastic in that firm's price.

This remark follows because a searcher visiting the lowest priced store will buy from that store regardless of whether the price induces him to search for further quotations.

In Carlson and McAfee (1983) and Stiglitz (1987), the rational-expectations assumption is crucial in ensuring that the firms have downward-sloping demand functions. Elastic demand allows each price to be an equilibrium response to the others. However, when there is a finite number of firms and searchers do not have rational expectations, Remark 6-2 suggests that at least one firm would have an incentive to slightly undercut the lowest price of the others. Pure-strategy equilibria are unlikely without the rational-expectations assumption. To give some structure to this claim, the example of Blind Search is considered.

6.3 An Example: Blind Search.

In addition to Remarks 6-1 and 6-2, Blind Search has the following two properties that enable definite statements to be made about equilibrium.

<u>Remark 6-3:</u>

In Blind Search, the recall option may be exercised.

As noted several times in previous chapters, this property is important when at least some consumers search more than one firm and there is a finite number of firms. Then, when two firms charge the same price, there is a positive probability that there will be a mass of consumers who visit both firms without visiting a lower priced store. This gives a profit function for each firm that is discontinuous at each of the other firms' prices.

Remark 6-4:

In Blind Search, the number of buyers buying from the highest priced firm is locally inelastic in that firm's price.

A firm which is the only one charging the highest price can only sell to those consumers for whom that price is their only quotation—that is, to the consumers who sample that firm first and then stop. Remark 6-1 shows that the number of consumers visiting the firm is independent of its price. It is a feature of Blind Search that the decision to seek a second quotation by those consumers is also independent of the price. An implication of Remark 6-4 is that a unique highest price must be the monopoly price.[2]

The independence of the first quotation (on the decision of a searcher to take a second) is the most important equilibrium property of Blind Search. Let h(1) denote the (exogenous) proportion of consumers who stop search after one quotation. In equilibrium search models, this value, whether exogenous or endogenous, has been crucial to modeling equilibrium price dispersion as it affects whether the maximum and minimum prices in a distribution can be the result of profit maximizing decisions. An equilibrium price-dispersion model has to explain how demand can be positive at the highest price, and how demand

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[2] In Chapter 5, Blind Search was presented as expenditure-minimizing search for a single unit of the good. If all consumers have the same choke price, above which no quantity is demanded, then it would be the monopoly price. As noted in Section 5.2, p 82, Blind Search also admits the interpretation of indirect-utility-maximizing search. This could give a downward-sloping demand function for the highest-priced firm, with the monopoly price where marginal revenue equals marginal cost.

can be greater at the lowest price so that both prices are Nash responses by firms. If the model can sustain the extreme prices in equilibrium, it is fairly easy to create conditions where intermediate prices are profit-maximizing choices by firms.

The importance of h(1) in allowing extreme prices is illustrated in the proof to the following theorem.

<u>Theorem 6-1:</u>

If there are a finite number of firms, firms have constant marginal costs, and consumer demand is inelastic, then Blind Search allows no price dispersion in equilibrium.

<u>Proof:</u>

Consider three cases for the value of h(1):

a) h(1) = 1 (that is, everyone visits only one firm).

This produces the famous Diamond (1971) result. Since every firm is a perfect monopolist (by Remark 6-4), the only equilibrium has every firm charging the monopoly price.

b) h(1) = 0 (that is, everyone visits at least two firms).

In this case, there can not be only one firm charging the highest price as its demand would be zero. If two or more firms charge the highest price, and that price exceeds marginal cost, there is always an incentive for firms to chisel when prices are identical. The only possible equilibrium is that of perfect competition:

$$p_1 = MC_1 = p_j = MC_j \qquad \forall\ j.$$

c) 0 < h(1) < 1.

Remark 6-2 implies that p_1 will always be arbitrarily close to p_2. In that case, the discontinuity in firm 2's profit function, implied by Remark 6-3 would leave it an incentive to undercut firm 1 unless $p_2 = MC_2$. With constant costs, price equaling marginal cost would imply zero profit. But positive profit can be earned with $p_2 > MC_2$ because some demand at the monopoly price is ensured by $h(1) > 0$, so $p_2 = MC_2$ is not an equilibrium response. There is therefore no equilibrium with $0 < h(1) < 1$.

□

If any of the conditions in the theorem is violated, the theorem does not hold. First, increasing marginal cost would allow positive profit at $p_2 = MC_2$. Second, if consumer demand is elastic (and $MC_2 > MC_1$), the profit-maximizing price for firm 1 need not be arbitrarily close to firm 2's. Finally, if there is a continuum of firms, there is zero probability of any two firms being sampled by the same consumer and so the discontinuity noted in Remark 6-3 does not apply.[3] These examples are special cases though. Blind Search in a Nash/ Bertrand framework with a finite number of firms does not generally allow existence of equilibrium with price dispersion. indeed, if $0 < h(1) < 1$, degenerate equilibria will also be rare.

It was suggested in Chapter 5 that Blind Search is useful for making comparisons with Rational Expectations Search. Here, the specific predictions of Blind Search led to Remarks 6-3 and 6-4 and hence to Theorem 6-1. Although these remarks do not necessarily hold in all adaptive search models, they do reflect general properties of these models. For instance, recall of previous prices will result when additional information about the price distribution makes a previously rejected price more valuable than it first appeared relative to an unknown draw from the distribution.

Remark 6-4, in particular, illustrates the way that Blind Search and

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[3] Is is also true that when, there is an infinity of firms, the distinction between models with and without rational expectations given by Remark 6-1 is lost, since in neither case can a single firm affect the price distribution.

Rational Expectations are two extremes in the amount of information available to consumers. The less information consumers have about the price distribution before they search, the more they learn about the mean of the distribution, and the less about the other moments, from the first quotation. But it is the higher moments which determine the benefit of continuing search. With adaptive search, then, the first quotation is less likely to affect the decision to seek a second than with rational-expectations search, and so the demand function will be less elastic at the highest price. In the limiting case of Blind Search, consumers have no information about prices; the first quotation gives no information about the variance of the distribution and so the demand curve is locally inelastic at the highest price.

6.4 Approximating Adaptive Search.

The example of the previous section illustrated some of the problems that can arise when modeling equilibrium without rational-expectations search, but did not show any way of avoiding the problems. Even in the case where there is a continuum of firms, so that Theorem 6-1 does not apply, the complexity of the stopping rule in Blind Search makes it difficult to derive tractable demand functions for firms.

The difficulty arises, not from the absence of rational expectations itself, but from the fact that stopping rules in adaptive search may depend on the entire sequence of observed prices. The two search models from which explicit demand functions have been derived in the equilibrium-search literature—with-replacement, non-adaptive sequential search, and FSS search—can both be viewed as special cases of a more general sequential model where the decision on whether to take another quotation depends on a single variable. In the former, of course, the stopping decision depends only on the minimum price sampled. FSS search can also be interpreted as a form of sequential decision making where the the searcher is precluded from observing the prices sampled before stopping (or from using the information contained in the prices) and so the stopping rule depends only on the number of firms sampled. As shown in Chapter 4, when sequential sampling is without replacement, the stopping rule depends on both the minimum price and the number of firms already sampled. The complexity of the optimal policy being described by two state variables rather than one, was enough to prevent the derivation of explicit demand functions.

Since adaptive-search models involve the beliefs about the price distribution changing as search progresses, the stopping rule in these models will always depend on the number of firms sampled and at least one other statistic. For the reasons discussed above, then, equilibrium modeling is only likely to be feasible with non-adaptive search. If adaptive search is the appropriate consumer behaviour to assume in a particular equilibrium model, it would seem that a non-rational-expectations version of either simple sequential search or FSS search is needed. Which of the two is best, depends on which most closely approximates the equilibrium results of adaptive search.

It does not follow that, because adaptive search is sequential, the best approximation to it is sequential. As suggested above, FSS search can be interpreted as a form of sequential strategy. It mimics adaptive search in its dependence on the number of firms sampled to determine the stopping rule. The basic sequential model ignores the sample size. It was because of the potential sub-optimality of sampling too many firms that Gastwirth (1976) incorporated the sample size into his naive truncated rule, referred to in Chapter 5.

To capture the properties of Blind Search in a tractable model, the best model to use is non-adaptive, non-rational-expectations, without-replacement FSS search. It is identical to Blind Search as far as the crucial number h(1) is concerned. Remarks 6-3 and 6-4 both apply to this model, and hence so does Theorem 6-1. Neither of these properties hold with non-adaptive sequential search. As noted earlier, the number of consumers buying from the first firm visited will be less elastic in that firm's price in adaptive-search models compared to non-adaptive sequential search models. This suggests that FSS search is a good approximation for adaptive search in general.

Wilde and Schwartz (1979) use FSS search without rational expectations in their equilibrium model. They assume a continuum of firms so the difficulties of Theorem 6-1 do not apply. They cite Gastwirth (1976) as a justification for assuming FSS search. The discussion here provides another justification.

6.5 Concluding Remarks.

Schwartz and Wilde (1982), noting the enthusiasm of policy makers for regulation of informationally imperfect markets, addressed the question of whether equilibrium-search models were sufficiently developed to yield relevant policy conclusions about the best form of and need for regulation of such markets. They were particularly concerned that the rational-expectations assumption of most equilibrium models was giving inappropriate results.

The main point of this chapter has been to show that many of the equilibrium results of search models depend entirely on the assumption of rational expectations, but that it is difficult to get results of any sort if that assumption is abandoned. The previous section suggested a way to avoid the complexities of adaptive search while approximating its main equilibrium properties. It does not, however, solve the problem of non-existence of equilibria when firm numbers are finite. If questions such as those interesting Schwartz and Wilde are to be addressed, more robust equilibrium-search models are needed.

Part C: An Equilibrium Model

Chapter 7: Inefficiency In Oligopoly When Price Information Is Endogenous

7.1 Introduction.

This chapter is more directly about equilibrium than the four chapters of Part B. There, the connection between the assumptions made about how consumers search and the resulting demand functions was investigated. Some general results about the nature of possible equilibria were attainable by this approach. This chapter considers the role of search in equilibrium models from a different direction: The comparative static results obtained here from a simple equilibrium-search model allow a comparison with results from product-differentiation models that do not involve search.

Chapter 2 emphasized the point that equilibrium-search models have the structure of product differentiation. Each firm's product is differentiated from the others' by the varying amounts of information consumers have following a random search process. A feature of product differentiation is that it makes it feasible for firms to charge different prices in an equilibrium. The possibility of price dispersion suggests some analogies with other product differentiation models. Whenever there is price dispersion due to transactions costs separating markets (for instance with markets separated by location, time, or information) there may be unrealized gains from trade—inefficiency—and hence opportunities for arbitrage. An arbitrager can be thought of as a dealer who, through economies of scale or some natural advantage, can combine such markets by reducing transactions costs. (Without transactions costs of some sort, the law of one price would prevail and so arbitrage would never be observed.) Arbitrage profits come from the increased gains from trade available in the enlarged market.

Intuitively, therefore, arbitrage might be expected to reduce dispersion and increase efficiency in a price dispersion model. When markets are separated by information, however, rather than by a physical characteristic such as product quality or location, the degree of differentiation is endogenous to the model and can be affected by the addition of a firm to the market. For instance, Stiglitz (1987) has shown that increasing the number of firms in a market characterized by consumer search has two opposing effects on the amount of competition in the market. Each firm's share of the market is reduced, but, with more firms, each firm has less impact on the amount of consumer search and so faces a less elastic demand curve. Introducing an arbitrager to the market may therefore create more inefficiency than it eliminates since it reduces the incentive to search and so in effect increases the amount of differentiation between firms.

Although many authors have suggested the possibility of specialist dealers appearing in equilibrium-search models,[1] arbitrage appears not to have been modeled formally in this context.[2] This chapter uses a simple equilibrium model to examine the effect that firm numbers in general and arbitrage in particular have on prices and efficiency. It is shown that arbitrage, as modeled here, unambiguously reduces consumer welfare and is in general socially inefficient as well. While the model is a special case so that the strong welfare results do not necessarily carry over to more general structures, the source of the results—i.e. the endogeneity of product differentiation due to information—is common to most models of equilibrium search.

The model used here is a simplified version of Carlson and McAfee (1983). In the following section, the basic model is outlined with welfare results derived in Section 7.3. The results from this price-dispersion model without arbitrage provide a benchmark for comparison when arbitrage is introduced into the model in Section 7.4. A numerical example is presented in Section 7.5, and

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[1] See, for example, Stigler (1961) and Butters (1977).

[2] Manning and Paterson (1980) have modeled arbitrage with search but not in an equilibrium framework.

the final section contains some concluding remarks. Most of the algebraic derivations of results are confined to Appendix D.

7.2 The Model: No Arbitrage.

Let there be m firms who are potential sellers of a homogeneous good. Firms are denoted by the subscript j. Each firm can produce unlimited quantities of the good with constant marginal costs and no fixed cost. Marginal cost α_j can differ between firms.

Any firm producing positive output is considered to be "participating" in the market. Let there be n participating firms posting prices ordered from lowest to highest,

$$0 < p_1 \leq p_2 \dots \leq p_n,$$

with $n \leq m$. The subscripts $j \in \{n+1 \dots m\}$ denote the non-participating firms.

Each firm must decide whether or not to participate, and, if so, what price to set. Firms follow Nash behaviour with respect to the prices set by the other firms. They set their own prices to maximize expected profit given by

$$\Pi_j(p_j|\mathbf{p}_{-j},n) \equiv (p_j-\alpha_j)q_j$$

where $\mathbf{p}_{-j}$ is the (n-1)-vector of all prices excluding p_j, and q_j is the expected demand function facing firm j.

Let $r_j^n(\mathbf{p}_{-j})$ be the reaction function of firm j so

$$r_j^n(\mathbf{p}_{-j}) = \underset{p_j}{\operatorname{argmax}}\left\{\Pi_j(p_j|\mathbf{p}_{-j},n\right\}.$$

The following definition describes a conventional Nash equilibrium with the added restriction that each firm participates.

Definition 7-1:

An n-vector of prices $\mathbf{p}^*$ is a "participant equilibrium" if and only if

a) $p_j^n = r_j^n(\mathbf{p}_{-j}^*)$ $\quad j = 1\ldots n,$

b) $q_j > 0$ $\quad j = 1\ldots n.$

To extend this definition to include all potential firms, it is also required that none of the non-participating firms has an incentive to enter taking the participating firms prices as given.

Definition 7-2:

An n-vector of prices $\mathbf{p}^*$ is an "entry equilibrium" if and only if

a) $\mathbf{p}^*$ is a participant equilibrium,

b) $\max_{p_j}\left\{\Pi_j(p_j|\mathbf{p}^*, n+1)\right\} \leq 0$ $\quad j = n+1\ldots m.$

By enabling the number of firms to be endogenously determined, the entry equilibrium gives a complete characterization of the market. The less general participant equilibrium will be used to analyze the welfare effects of entry.[3]

There are M consumers who each demand one unit of the good. They are assumed to know the distribution of prices but search to find the prices charged by particular firms. Search is sequential, with replacement. Consumers only sample from participating firms; the probability of any consumer sampling any participating firm on a single search is $\frac{1}{n}$, independent of all other sampling by himself or other consumers.

[3] The model in this section differs from Carlson and McAfee's in only two respects: They do not model entry directly, but do allow upward-sloping marginal cost functions. A result of constant marginal costs is that nothing here depends on the number of consumers. This is not true of the Carlson and McAfee model.

Consumers differ only in their unit cost of search x. Let M be sufficiently large so that the distribution of x can be represented as a continuum. It is assumed that x is uniformly distributed over the range [0,T].

Define x_k as the expected gain to a consumer from taking one more price quotation when p_k is the lowest quotation already received.[4] Then,

$$x_k = \frac{1}{n}\sum_{i=1}^{k-1}(p_k-p_i)$$

$$= \frac{1}{n}\left(p_k(k-1) - \sum_{i=1}^{k-1} p_i\right) \qquad k=2..n, \tag{7-2-1}$$

$$x_1 = 0.$$

A firm charging p_j will sell nothing to consumers for whom $x < x_j$, will share with the first (j-1) firms the consumers for whom $x_j \le x < x_{j+1}$, and so on. Therefore, by the assumption of uniformly distributed costs,

$$q_j = \frac{M}{T}\sum_{k=j}^{n}\frac{1}{k}\left(x_{k+1}-x_k\right) \qquad j = 1..n, \tag{7-2-2}$$

where $x_{n+1} \equiv T$.

Define $p_0 \equiv 0$ and $p_{n+1} \equiv T+\bar{p}$ where $\bar{p} = \frac{1}{n}\sum_{i=1}^{n} p_i$ so that (7-2-1) is defined for k=1, n+1. Then (7-2-1) substituted into (7-2-2) gives

$$q_j = \frac{M}{nT}\sum_{k=j}^{n}\left(p_{k+1} - \frac{k-1}{k}p_k - \sum_{i=1}^{k}\frac{p_i}{k} + \sum_{i=1}^{k-1}\frac{p_i}{k}\right).$$

Now $$\sum_{k=j}^{n} p_{k+1} = \sum_{k=j}^{n} p_k + p_{n+1} - p_j$$

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[4] In the notation of Chapters 3 and 4, x_k is the dual, $x_k=C(p_k)$. The notation here is Carlson and McAfee's.

so
$$q_j = \frac{M}{nT}\left(\sum_{k=j}^{n}(p_k - \frac{k-1}{k}p_k - \frac{1}{k}p_k) + p_{n+1} - p_j\right)$$
$$= \frac{M}{nT}(p_{n+1} - p_j)$$
$$= \frac{M}{nT}(T+\bar{p}-p_j) \qquad j=1..n. \tag{7-2-3}$$

The first-order conditions for profit maximization are

$$\frac{\partial \Pi_j}{\partial p_j^*} = q_j + (p_j^* - \alpha_j)\frac{\partial q_j}{\partial p_j} = 0. \tag{7-2-4}$$

Substituting (7-2-3) into (7-2-4) gives the reaction functions

$$r_j^n(\mathbf{p}^*) = \frac{1}{2(n-1)}\left(nT + (n-1)\alpha_j + \sum_{i\neq j} p_i\right) \qquad j=1..n. \tag{7-2-5}$$

From (7-2-4),

$$\frac{\partial^2 \Pi}{\partial p_j^2} = \frac{-2(n-1)M}{n^2 T} < 0$$

so that the second-order conditions for profit maximization are satisfied.

Equations (7-2-5) can be solved for an explicit equilibrium,

$$p_j^* = \frac{n}{n-1}T + \frac{n}{2n-1}\bar{\alpha}^n + \frac{n-1}{2n-1}\alpha_j \qquad j=1..n \tag{7-2-6}$$

where $\bar{\alpha}^n = \frac{1}{n}\sum_{i=1}^{n}\alpha_i$.

The solution is confirmed by substitution of (7-2-6) back into (7-2-5). For ease of reading, the superscript on $\bar{\alpha}$ denoting the number of firms will be omitted where no ambiguity results from doing so.

The set of prices p_j^* given by (7-2-6) are profit maximizing for individual firms. To complete the description of equilibrium, the conditions on entry and exit in Definitions 7-1 and 7-2 need to be satisfied.

Without loss of generality, let the non-participating firms be labeled so that $\alpha_{n+1} \le \alpha_{n+2} \cdots \le \alpha_m$.

Proposition 7-1:

a) A necessary and sufficient condition for equations (7-2-6) to describe a participant equilibrium is

$$\alpha_n - \bar{\alpha} < \frac{2n-1}{n-1}T. \tag{7-2-7}$$

b) To describe an entry equilibrium, it is also necessary that

$$\alpha_{n+1} - \bar{\alpha} \ge \frac{2n^2-1}{n(n-1)}T. \tag{7-2-8}$$

Proof:

From the first-order conditions and equation (7-2-3),

$$q_j = \frac{M(n-1)}{n^2 T}(p_j^* - \alpha_j)$$

and so $\quad q_j > 0 \quad \text{iff} \quad (p_j^* - \alpha_j) < 0$

$$\text{iff} \quad \alpha_j - \bar{\alpha} < \frac{2n-1}{n-1}T \qquad j = 1 \ldots n. \tag{7-2-9}$$

Recall that the labeling of participating firms is not arbitrary but based on the ordering of prices. It follows trivially from (7-2-6) that $p_i \lesseqqgtr p_j$ as $\alpha_i \lesseqqgtr \alpha_j$ so that

$$\alpha_1 \le \alpha_2 \le \ldots \alpha_n.$$

Therefore, for all firms to have positive output, it is necessary and sufficient that (7-2-9) holds for j=n.

Also from the first-order conditions, the expected demand of an entering firm j, j>n, when the prices of the other participating firms are given is

$$q_j = \frac{Mn}{(n+1)^2T}(r_j^{n+1}(p^*)-\alpha_j),$$

so $$\max\left\{\Pi_j(p_j|p^*,n+1\right\} \le 0 \quad \text{iff} \quad (r_j^{n+1}(p^*)-\alpha_j) \le 0.$$

Substituting (n+1) for n in equation (7-2-5), this gives

$$\left((n+1)T - n\alpha_j + \sum_{i=1}^{n} p_i^*\right) \le 0$$

$$\Rightarrow \quad \alpha_j-\bar{\alpha} \ge \frac{2n^2-1}{n(n-1)}T \qquad j = n+1 \;\ldots\; m$$

(using (7-2-6)).

The proposition follows from the ordering of non-participating firms.

□

In a perfectly competitive market, all participating firms must have access to the most efficient technology or be forced from the market. Proposition 7-1 is a similar result, stating that it will not be profitable for firms to produce if their production costs are too far above the market average relative to search costs. The following corollary demonstrates another similarity with technological efficiency.

<u>Corollary:</u>

In any entry equilibrium, the n participating firms will be the n-most efficient.

Proof:

From (7-2-7) and (7-2-8),

$$\alpha_j-\bar{\alpha} \;\geq\; \frac{2n^2-1}{n(n-1)}T \;>\; \frac{2n^2-n}{n(n-1)}T \;>\; \alpha_n-\bar{\alpha} \qquad j = n+1\ldots m$$

so $$\alpha_j > \alpha_n \qquad j = n+1\ldots m.$$

□

In the remainder of this section, the existence and uniqueness of entry equilibria are considered. The corollary limits the number of possible equilibria as it implies that there is at most one entry equilibrium with n participating firms for any n.

Proposition 7-2:

If the number of potential firms exceeds one, there is no entry equilibrium with only one participating firm.

Proof:

If n=1, then in (7-2-3) $p_1=\bar{p}$ and $q_1=M$, and so a monopolist could raise its price indefinitely without affecting demand. Thus a potential firm taking the monopolist's price as given would always have an incentive to enter.[5]

□

Proposition 7-3:

For any participant equilibrium where $n \geq 2$ and $\alpha_n < \alpha_{n+1}$, if the (n+1)th firm has an incentive to enter, then the resulting participant equilibrium with n+1 participating firms exists.

---

[5] This follows because of the assumption that consumer demand is inelastic everywhere. More realistically, of course, consumers would have a "choke price" above which they would not purchase. However, as a choke price is not necessary to set a limit on prices when there is more than one firm, modeling it explicitly only increases the algebraic manipulation required without adding to the intuition.

Proof:

The result would fail to hold only if the (n+1)th firm had an incentive to enter at the existing prices but could no longer compete once the other firms adjusted their prices to the new competition.

From (7-2-8), the (n+1)th firm has an incentive to enter if

$$\alpha_{n+1}-\bar{\alpha}^{n} < \frac{2n^2-1}{n(n-1)}T$$

$$\Rightarrow \quad \frac{n}{n+1}(\alpha_{n+1}-\bar{\alpha}^{n}) < \frac{2n^2-1}{(n+1)(n-1)}T$$

$$\Rightarrow \quad \alpha_{n+1}-\bar{\alpha}^{n+1} < \frac{2n^2-1}{(n+1)(n-1)}T$$

$$\Rightarrow \quad \alpha_{n+1}-\bar{\alpha}^{n+1} < \frac{2n+1}{n}T \qquad (7\text{-}2\text{-}10)$$

(if $n \geq 2$).

But this is the condition for existence of a participant equilibrium with n+1 firms, given by substituting n+1 for n in (7-2-7).

□

Proposition 7-3 shows that an entrant firm that is only marginally profitable at the existing prices will *benefit* from the other firms' reacting to its presence. This counter-intuitive result is an indication of the anti-competitive effect that entry can have by introducing more noise into the market.

Proposition 7-4:

a) If there exists any participant equilibrium with at least two firms, then there will always be an entry equilibrium with at least as many participating firms.

b) There may be several entry equilibria.

Proof:

Note that if there is any participant equilibrium with n firms, then the existence condition (7-2-7) implies that there is a participant equilibrium with the n-most efficient firms participating. Also note that a participant equilibrium with all m potential firms participating must be an entry equilibrium.

The proof of a) is by construction. Consider the smallest value of n for which there is a participant equilibrium with the n-most efficient firms, $n \geq 2$. Either it is an entry equilibrium or the (n+1)th firm has an incentive to enter. If the latter is true, then Proposition 7-3 implies that there is a participant equilibrium with the (n+1) most efficient firms. It then must either be an entry equilibrium or the (n+2)th firm has an incentive to enter and so on. Clearly, by adding firms in order of costs in this way, an entry equilibrium must eventually be reached.

The possible multiplicity of equilibria results from the fact that an entering firm which is only marginally profitable creates noise that is beneficial to it. Consider a participant equilibrium with n firms where

$$\frac{2n^2-1}{n(n+1)}T \quad < \quad (\alpha_{n+1}-\bar{\alpha}^n) \quad < \quad \frac{2n+1}{n}T$$

From the proof to Proposition 7-3, the first inequality implies that there is an entry equilibrium with n firms as the (n+1)th firm has no incentive to enter, and the second shows that there would be a participant equilibrium with n+1 firms and hence an entry equilibrium with at least that many.

□

The non-uniqueness result does not depend on the assumption that potential entrants treat the existing prices as given. Even if they were to allow for the adjustment of other firm's prices when considering entry, there could still be a multiplicity of equilibria. This is easily shown. From (7-2-7), there will be a participant equilibrium after the entry of the (n+1)th firm only if

$$\frac{n}{n+1}(\alpha_{n+1}-\bar{\alpha}^{n}) < \frac{2n+1}{n}T$$

However, if two firms with cost α_{n+1} enter simultaneously, the condition is

$$\frac{n}{n+2}(\alpha_{n+1}-\bar{\alpha}^{n}) < \frac{2n+3}{n+1}T$$

which is less restrictive.

The interpretation of this result is that although it may not pay for a single firm to enter, two firms may be able to jointly create enough noise in the market to make it profitable for both to do so!

It is clear from these results, that entry will not always increase the competition in a market. This feature of the model is now developed more fully.

7.3 Welfare Effects of Entry.

The characterization of entry equilibria in the previous section were positive results giving conditions for when entry would occur. In this section, the more normative question of whether entry is beneficial to consumers and socially efficient is considered. Accordingly, participant rather than entry equilibria are compared. It is simply assumed, therefore, that the existence condition (7-2-7) holds in all cases, and that there are always at least two firms participating so that prices are bounded.

Before comparisons of consumer welfare and efficiency in different equilibria are made, it will be necessary to derive measures for these concepts. This is now done.

In the case of perfect competition—i.e. T=0, and all firms having access to the most efficient technology—every consumer would pay α_1 for the good and incur no search costs. Therefore a simple measure of the cost to the consumer of imperfect information is the difference between the total average expected payment per consumer and α_1. This is given the term "consumer costs" denoted CC.

Consumer costs can be decomposed into 3 parts:

i) the average expected expenditure on search per consumer (ASE),

ii) the average expected unit production cost in excess of α_1 (APC),

iii) the average expected markup by firms on unit costs (AM).

Measures ii) and iii) are averages weighted by market-share so that all costs are "per-consumer" averages.

The term "social costs", denoted SC, is used to describe the inefficiency of imperfect information in a Pareto sense. This is defined as average search costs plus excess production costs. The assumption of inelastic consumer demand implies that the monopoly markup is simply a lump-sum transfer from consumers to firms, with no deadweight loss.

Expressions for these costs, derived in Appendix D, are

$$\text{ASE} = \frac{1}{2T}\text{var } p \tag{7-3-1}$$

$$\text{APC} = \sum_{j=1}^{n} w_j \alpha_j - \alpha_1 \tag{7-3-2}$$

$$\text{AM} = \sum_{j=1}^{n} w_j (p_j - \alpha_j) \tag{7-3-3}$$

where w_j are the market-share weights

$$w_j = \frac{1}{\sum_{i=1}^{n} q_i} q_j$$

Substituting in the values of p_j and q_j from (7-2-6) and (7-2-4), (7-3-1) - (7-3-3) can be reexpressed

$$ASE = \frac{(n-1)^2}{2T(2n-1)^2}\text{var } \alpha \tag{7-3-4}$$

$$APC = (\bar{\alpha}-\alpha_1) - \frac{n-1}{T(2n-1)}\text{var } \alpha \tag{7-3-5}$$

$$AM = \frac{n}{n-1}T + \frac{n(n-1)}{T(2n-1)^2}\text{var } \alpha \tag{7-3-6}$$

giving $CC = ASE + APC + AM$

$$= \frac{n}{n-1}T + (\bar{\alpha}-\alpha_1) - \frac{(n-1)^2}{2T(2n-1)^2}\text{var } \alpha \tag{7-3-7}$$

and $SC = ASE + APC$

$$= (\bar{\alpha}-\alpha_1) - \frac{(3n-1)(n-1)}{2T(2n-1)^2}\text{var } \alpha \tag{7-3-8}$$

These measures are used to compare a participant equilibrium with (n-1) firms to one with n firms with the "entrant" denoted by the subscript e. The superscripts n-1 and n are used to distinguish variables from the two equilibria.

In oligopoly models with no uncertainty, whether perfect competition is achieved with any number of firms greater than one, or only achieved in the limit as the number of firms tends to infinity, depends on whether firms make a Bertrand or Cournot conjecture. Although search models imply that firms are Bertrand price setters, the downward sloping demand functions that firms face resemble Cournot. In the following special case, the Cournot result is approximated.

Theorem 7-1:

If all firms have the same production costs, then consumer costs are lower the greater is the number of firms.

Proof:

From (7-3-4) - (7-3-6), var $\alpha = 0$ implies that there are no search costs or production inefficiencies, while the monopoly markup for n firms = $\frac{n}{n-1}T$, which is a declining function in n.

□

As outlined in Section 7.1, entry has two opposing effects on competitiveness in a market; market-share is reduced, but so is the incentive for consumers to search. By this is meant the *ex-ante* incentive to search at the existing prices. For a given distribution of prices lower than the latest one sampled by a consumer, the probability of sampling one of these prices is reduced the greater is the number of firms, and so the marginal benefit of taking another quotation is lower. Theorem 7-1 shows that when firms are identical, the effect of reduced market share dominates. When there is price dispersion, however, so that consumers do search, the search effect becomes more important. This is demonstrated by the next theorem. The delta, Δ, is used to indicate the difference in a variable as a result of entry. So $\Delta CC = CC^n - CC^{n-1}$ etc.

Theorem 7-2:

When there is price dispersion, the entry of an additional firm will:

a) reduce consumer costs if and only if the entrant's production cost is less than some critical value—i.e. there exists α_e^c such that $\Delta CC \lesseqqgtr 0$ as $\alpha_e \lesseqqgtr \alpha_e^c$;

b) reduce social costs if and only if the entrant's production cost is less than some critical value below the market average—i.e. there exists $\alpha_e^s < \overline{\alpha}^{n-1}$ such that $\Delta SC \lesseqgtr 0$ as $\alpha_e \lesseqgtr \alpha_e^s$;

c) always benefit consumers if there is an increase in efficiency—i.e. $\Delta SC < 0 \Rightarrow \Delta CC < 0$;

d) cause a greater increase in consumer and social costs, the greater is the existing amount of price dispersion—i.e. $\dfrac{\partial \Delta CC}{\partial \text{var } p} > 0$ and $\dfrac{\partial \Delta SC}{\partial \text{var } p} > 0$;

e) increase the range and variance of *existing* firms' prices.

Proof:

The proof to this theorem, which involves predictable algebraic manipulation of equations (7-3-7) and (7-3-8), has no economic content and so is relegated to Appendix D.

□

The fifth result in Theorem 7-2 represents the effect of entry on search isolated from that on market-share. By reducing the *ex ante* incentive to search, entry results in a shift in relative market share from the lower to higher priced firms of those already in the market, as well as an absolute shift to the entrant. The increased dispersion of existing prices results from the shift in relative market-share, and mitigates in part the *ex post* incentive to search.

7.4 Arbitrage.

Now let one of the n firms be an arbitrager, denoted by subscript a, who, instead of producing the good, buys from one firm and sells to consumers at a higher price. It is assumed that the arbitrager can only buy from the lowest cost firm[6] and that he has an additional constant marginal cost α_a in selling

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[6] In an earlier version of this chapter, I assumed that the arbitrager purchases from the lowest *priced* firm. The less intuitive assumption of purchasing from the lowest cost firm is used here purely to simplify the

the good. The arbitrager is as likely as any other firm to be searched by any consumer. The subscripts 1 and n are still used to denote the lowest and highest cost non-arbitraging firms respectively even though it will be possible that $p_2 < p_1$ and $p_a > p_n$ in equilibrium.

The interest here is in the welfare implications of arbitrage and so the wider question of entry is ignored. An arbitrage equilibrium in defined in the same way as a participant equilibrium in Section 7.2, the latter term only being used to refer to equilibria without arbitrage.

Definition 7-3:

An n-vector of prices $\mathbf{p}^*$ is an "arbitrage equilibrium" if and only if

a) $p_j^* = r_j^a(\mathbf{p}_{-j}^*)$

b) $q_j > 0 \qquad \forall\ j$

where $r_j^a(\mathbf{p}_{-j})$ are the reaction functions

$$r_j^a(\mathbf{p}_{-j}) = \underset{p_j}{\operatorname{argmax}}\left\{(p_j-\alpha_j)q_j\right\} \qquad j\neq a$$

$$r_a^a(\mathbf{p}_{-a}) = \underset{p_a}{\operatorname{argmax}}\left\{(p_a-\alpha_a-p_1)q_a\right\}$$

The first-order conditions for profit maximization are

$$q_j + (p_j-\alpha_j)\frac{\partial q_j}{\partial p_j} = 0 \qquad j\neq a \qquad (7\text{-}4\text{-}1)$$

---

presentation. When the arbitrager buys from the lowest priced firm there is a discontinuity in the demand functions for each firm at the lowest of the other firms' prices leading to some very messy existence conditions. Since one of these conditions requires that the arbitrager buy from the lowest cost firm, the results are exactly the same with both interpretations. The example in Section 7.5 is valid for both.

$$q_a + (p_a - \alpha_a - p_1)\frac{\partial q_a}{\partial p_a} = 0 \tag{7-4-2}$$

This model differs from that of Section 7.2 only in the interaction between the arbitrager and the lowest cost firm. As all firms appear identical to consumers, the demand functions are the same as in (7-2-3) with the exception that the arbitrager's demand is also demand for firm 1. So

$$q_j = \frac{M}{nT}(T+\bar{p}-p_j) \qquad j \neq 1 \tag{7-4-3}$$

$$q_1 = \frac{M}{nT}(T+\bar{p}-p_1) + q_a \tag{7-4-4}$$

Substituting into the first-order conditions gives the reaction functions,

$$r_a^a(\mathbf{p}_{-a}) = \frac{1}{2(n-1)}\left(nT + (n-1)\alpha_a + \sum_{i\neq 1,a} p_i + np_1\right) \tag{7-4-5}$$

$$r_1^a(\mathbf{p}_{-1}) = \frac{1}{2(n-2)}\left(2nT + (n-2)\alpha_1 + 2\sum_{i\neq 1,a} p_i - (n-2)p_a\right) \tag{7-4-6}$$

$$r_j^a(\mathbf{p}_{-j}) = \frac{1}{2(n-1)}\left(nT + (n-1)\alpha_j + \sum_{i\neq j} p_i\right) \tag{7-4-7}$$

This system of n equations has the solution, shown by substitution back into the reaction functions,

$$p_j = \frac{1}{(n-2)(2n-1)}\left((n-1)(5n-2)\phi + 2n(n-1)\alpha_a + n(n-2)\alpha_1 + (n-1)(n-2)\alpha_j\right) \tag{7-4-8}$$

$$p_a = \frac{1}{(n-2)}\left(4(n-1)\phi + 2(n-1)\alpha_a + (n-2)\alpha_1\right) \tag{7-4-9}$$

$$p_1 = \frac{1}{(n-2)}\left((3n-2)\phi + n\alpha_a + (n-2)\alpha_1\right) \tag{7-4-10}$$

where $$\phi \equiv \frac{1}{5n^2-6n+2}\left(n(2n-1)T + (n-1)^2(\bar{\alpha}^a-\alpha_1) - (2n^2-2n+1)\alpha_a\right). \tag{7-4-11}$$

The average cost $\bar{\alpha}^a$ is the average for the non-arbitraging firms,

$$\bar{\alpha}^a = \frac{1}{n}\sum_{j\neq a}\alpha_j$$

Again the superscript will be omitted where possible.

Existence requires positive output for all firms. Conditions are given by

Proposition 7-5:

For equations (7-4-8)-(7-4-10) to describe an arbitrage equilibrium it is necessary and sufficient that

a) $\phi > 0$ (7-4-12)

b) $(n-1)(5n-2)\phi + 2n(n-1)\alpha_a > n(n-2)(\alpha_n-\alpha_1)$ (7-4-13)

Proof:

The first-order conditions, (7-4-1) and (7-4-2), guarantee that outputs (and hence profits) are positive if the markups on marginal costs are positive. That is,

$$q_j > 0 \quad \text{iff} \quad (p_j-\alpha_j) > 0 \qquad \forall\ j\neq a$$

$$q_a > 0 \quad \text{iff} \quad (p_a-\alpha_a-p_1) > 0$$

From (7-4-9) and (7-4-10), ϕ is just the unit profit of arbitrage

$$\phi = (p_a-\alpha_a-p_1),$$

so $\phi > 0 \Leftrightarrow q_a > 0$

and from (7-4-10)

$$(p_1-\alpha_1) = \frac{1}{n-2}\Big((3n-2)\phi + n\alpha_a\Big)$$

so $\phi > 0 \Rightarrow q_1 > 0.$

This has the interpretation that, if arbitrage is profitable, selling to the arbitrager must be profitable.

From (7-4-8),

$$(p_j-\alpha_j) = \frac{1}{n-2}\Big((n-1)(5n-2)\phi + 2n(n-1)\alpha_a - n(n-2)(\alpha_j-\alpha_1)\Big) \qquad (7\text{-}4\text{-}14)$$

so $(p_n-\alpha_n) > 0$ iff (7-4-13) holds. By assumption,

$$\alpha_n \geq \alpha_j \quad \forall\ j{\neq}1,a$$

so from (7-4-14),

$$(p_n-\alpha_n) > 0 \Rightarrow (p_j-\alpha_j) > 0 \quad \forall\ j \neq 1,a.$$

□

In the non-arbitrage case, firms benefit from other firms having high costs and can only participate if their own costs are not too far in excess of the market average relative to search costs. Similar results are found here. For the arbitrager, from (7-4-11), a higher spread of search costs, T, implies that participation can be sustained with higher relative costs, but that it suffers from high costs in its supplying firm. For the firms not involved in arbitrage, using (7-4-11) to substitute out ϕ in (7-4-14),

$$(p_j-\alpha_j) > 0$$

$$\text{iff} \qquad n(2n-1)(n-1)(5n-2)T + (n-1)(n-2)(2n-1)\alpha_a - n(n-2)(5n^2-6n+2)(\bar{\alpha}-\alpha_1) - (n^3-7n^2+7n-2)(\alpha_j-\bar{\alpha}) > 0.$$

Again, higher search costs imply that participation can be sustained with higher relative costs. The additional terms, α_a and $(\bar{\alpha}-\alpha_1)$, have opposite signs from (7-4-11) indicating the competition from the arbitrager.

In considering the welfare implications of arbitrage, the comparison between an n-firm arbitrage equilibrium and the corresponding (n-1)-firm participant equilibrium is of particular interest; that is, the effect of introducing an arbitrager into a market with n-1 firms.

As in Section 7.3, the effect of entry rather than whether entry will occur is being considered, so it is assumed that the existence conditions, (7-2-7) and (7-4-12) - (7-4-13), hold where appropriate, and that there are at least two non-arbitraging firms so that prices are bounded.

The impact on prices can be easily stated. Using superscripts n-1 and a to distinguish between variables from the equilibria before and after the entry of the arbitrager,

$$p_j^{n-1} = \frac{1}{(n-2)(2n-3)}\Big((n-1)(2n-3)T + (n-1)(n-2)\bar{\alpha} + (n-2)^2\alpha_j\Big)$$

(from substituting (n-1) for n in equations (7-2-6))

$$= \frac{1}{(n-1)(n-2)(2n-3)}\Big((n-2)(5n^2-6n+2)\phi - (2n^2-6n+3)T + (n-2)(2n^2-2n+1)\alpha_a + (n-2)(n-1)^2\alpha_1 + (n-1)(n-2)^2\alpha_j\Big)$$

(using (7-4-11)),

Therefore,

$$p_1^a-p_1^{n-1} = \frac{1}{(n-1)(n-2)(2n-3)}\Big((2n^2-6n+3)T + (n^2-2n+2)\alpha_a + (n^3-3n^2+5n-2)\phi\Big) \qquad (7\text{-}4\text{-}15)$$

$$p_j^a - p_j^{n-1} = \frac{1}{(n-1)(n-2)(2n-1)(2n-3)}\Big((2n-1)(2n^2-6n+3)T + (3n-2)\alpha_a + (n-1)(n-2)(\alpha_j-\alpha_1) - (2n^3-10n^2+9n-2)\phi\Big) \quad (7\text{-}4\text{-}16)$$

Equation (7-4-15) confirms the intuitive result that the lowest price must rise as a result of arbitrage, but from (7-4-16), all other prices are likely to rise as well. Indeed,

Theorem 7-3:

The introduction of an arbitrager into a market with price dispersion will increase the dispersion in the prices of the firms not selling to the arbitrager

Proof:

Consider two firms $i, j \neq 1, a$ where $\alpha_j > \alpha_i$ so that $p_j^{n-1} > p_i^{n-1}$. From equation (7-4-16),

$$(p_j^a - p_i^a) - (p_j^{n-1} - p_i^{n-1}) = \frac{1}{(2n-1)(2n-3)}(\alpha_j - \alpha_i) \quad > 0$$

□

The effect on the range of prices that consumers face is ambiguous; it is not necessary that $p_1 < p_2$ or $p_a < p_n$ and so the firms charging the extreme prices may change after the arbitrager enters. It is possible for the perverse result to occur that arbitrage increases the range. The example in Section 7.5 confirms this possibility.

Theorem 7-3 is a very similar result to Theorem 7-2 e). This suggests that it is the increase in the number of firms and the resulting effect on the incentive to search at the previous prices that is driving the the price movements and not the particular properties of arbitrage. To separate out these two effects when considering welfare, Theorem 7-4 considers the effect of a firm switching from production to arbitrage while the numbers of firms is kept constant. Theorem 7-5 then combines these results with the effects of firm numbers derived in Theorem 7-2 to give the overall effect on welfare of the entry of an arbitrager.

When using the measures CC and SC to describe an arbitrage equilibrium, two qualifications to equations (7-3-2) and (7-3-3) are needed. First, the total cost of production of one unit sold by the arbitrager is $(\alpha_a+\alpha_1)$ so this is used in place of α_a; and second, to avoid double counting, the market-share weight w_1 is calculated from firm 1's sales to consumers only and not from those to the arbitrager.

The firm switching to arbitrage will be denoted by the subscript e to facilitate comparison with Theorem 7-2. The results of this switch will obviously depend on how the firm's cost as a producer, α_e, compares to its cost as an arbitrager α_a. It is assumed that the overall marginal cost to the economy is kept constant, i.e. $\alpha_e=\alpha_a+\alpha_1$. This means that social costs would be unchanged if firms did not change their prices. The changes in welfare that result from the switch to arbitrage can then be attributed entirely to the *market* properties of arbitrage and not to the technology.

<u>Theorem 7-4:</u>

The switch of a firm e from production to arbitrage when $\alpha_a=\alpha_e-\alpha_1$ will:

a) cause a rise in all prices and hence reduce consumer welfare;

b) increase social welfare if and only the switching firm's cost is greater than some critical value above the average of the other firms—i.e. there exists $\alpha_e^a > \bar{\alpha}^a$ such that $\Delta SC \gtreqless 0$ as $\alpha_e \lesseqgtr \alpha_e^a$.

<u>Proof:</u>

The formal proofs of Theorems 7-4 and 7-5 are left for Appendix D but an intuitive sketch is given here.

The lowest cost firm's price increases because of the added demand of the arbitrager, as will the price of the firm switching to arbitrage because it has the cost of firm 1's markup in addition to $\alpha_1+\alpha_a$. The prices of all other firms rise as the cross-price effects from p_1 and p_a cause their demand to increase. Consumers choose their actions to minimize consumer costs. An increase in all prices must increase consumer costs even if the search cost component of the measure is reduced.

The ambiguous effect on social welfare arises because the transfer of demand away from a firm switching to arbitrage may be socially beneficial if that firm is an inefficient producer.

□

Theorem 7-5:

The addition of an arbitrager into a market will

a) always reduce consumer welfare;

b) reduce social welfare except possibly for very low α_a — i.e. if there is any value for α_a where arbitrage increases social welfare given T and $(\bar{\alpha}-\alpha_1)$, then there exists $\alpha_a^s < (\bar{\alpha}-\alpha_1)$ such that $\Delta SC \lesseqgtr 0$ as $\alpha_a \lesseqgtr \alpha_a^s$.

The intuition underlying the reduction in consumer welfare is in the relative importance of the two entry effects when there is arbitrage. The entry of a firm as an arbitrager allows all firms to benefit from the steepening of demand curves. The opposing effect of reduced market share is minimized, however, as the high p_a resulting from the arbitrager having p_1 not α_1 as a cost results in low demand for the additional firm. In this way, arbitrage provides a market structure which results in some of the gains from collusive behaviour being realized by non-cooperative firms. The arbitrage of this model is similar to the effect of firm 1 setting up a second outlet to nearly double the probability of being searched by any consumer.

Although it is possible to generate examples where introducing arbitrage increases social welfare, they require extreme values of parameters and, although it is not clear why, at least six firms including the arbitrager. The result that arbitrage will generally reduce efficiency *even if costless* (i.e. if $\alpha_a=0$) is contrary to the intuition on arbitrage. As suggested in the Section 7.1, this is largely because, by adding an extra firm, arbitrage creates more inefficiency than it eliminates. For instance, it could be profitable for an arbitrager to enter a market where there is no ex-ante price dispersion at all. This can be seen from equation (7-4-11). If all the non-arbitraging firms have the same costs (so that $\bar{\alpha}-\alpha_1 = 0$), the profitability condition for arbitrage ($\phi > 0$) can still hold.

The activity of the arbitrager who enters a market characterized by a single price is still arbitrage in the sense that it is enabling consumers to buy indirectly from an efficient firm when it is too costly for them to search enough to do so directly. It is only because of the presence of the arbitrager, however, that search is necessary at all. In this case the arbitrager's profit comes from arbitraging away some of the noise that it itself has created!

7.5 A Numerical Example.

To illustrate the results of the preceding sections, examples of three equilibria are given in this section:

a) A 2-firm participant equilibrium;

b) the corresponding 3-firm participant equilibrium with $\alpha_e = \bar{\alpha}$;

c) the corresponding 3-firm arbitrage equilibrium with $\alpha_a = \alpha_e - \alpha_1$.

a) Let $n=2$, $T=1.25$, $\alpha_1=1$, $\alpha_2=5$

then $p_1 = 4.833$ $p_2 = 6.167$ $\bar{p} = 5.5$

$\text{var } \alpha = 4.0$ $\text{var } p = 0.444$

b) $n=3$, $T=1.25$, $\alpha_1=1$, $\alpha_2=5$, $\alpha_e=3$

then $p_1 = 4.075$ $p_2 = 5.675$ $p_e = \bar{p} = 4.875$
$\text{var } \alpha = 2.667$ $\text{var } p = 0.427$

c) $n=3$, $T=1.25$, $\alpha_1=1$, $\alpha_2=5$, $\alpha_a=2$

then $p_1 = 7.181$ $p_2 = 7.534$ $p_a = 9.207$ $\bar{p} = 7.974$
$\text{var } p = 0.781$

In all cases, existence is shown by the equilibrium prices exceeding marginal cost.

Note that the range of prices facing the consumer has widened in Example c) compared to Example a), confirming the possibility suggested in Section 7.4.

The search, production, markup, consumer, and social costs of each equilibrium are given in Table 7-1.

Table 7-1.

	search	production	markup	CC	SC
a)	0.1778	0.9333	3.2111	4.3222	1.1111
b)	0.1707	1.1467	2.3870	3.7044	1.3173
c)	0.3123	1.8115	4.5381	6.6619	2.1238

The 3-firm non-arbitrage equilibrium is better for the consumer than the 2-firm although Pareto-inferior to it, while the arbitrage equilibrium is inferior to both by all criteria.

7.6 Concluding Remarks.

This chapter has demonstrated some counter-intuitive results from changing firm numbers and introducing arbitrage in an equilibrium-search model. Particularly surprising is the negative effect of arbitrage on consumer welfare and efficiency even when the effect from changing the number of firms is abstracted out, as in Theorem 7-4. The key to this result is the strong assumption that, to consumers, the arbitrager appears identical to the other firms. The monopoly power of firms exists because they are differentiated due to the search costs of consumers. Since the arbitrager also has to be searched by consumers, he does not reduce this transaction cost. The analogy with other classes of differentiated products models is interesting. In a model of locational separation, for instance, the analogous dealer would be one who located himself no closer to consumers than existing firms. Again, by not reducing the transactions costs of consumers, such a dealer would not increase the efficiency in the market. Unlike the present case, however, the dealer would not make a profit or *reduce* the efficiency.

Although the assumption that the arbitrager is no more likely to be searched than other firms is perhaps unrealistic, it does illustrate very strongly the difference between models where firms are differentiated by an exogenous characteristic and equilibrium search models where the differentiation is endogenous. It is true, however, that arbitrage will more readily perform its function of increasing the gains from trade and hence be more profitable in market structures where the arbitragers are more likely to be searched by consumers. Arbitrage is therefore more likely to be present where there are such structures. Retailing, where a small number of chains deal with large numbers of manufacturers and consumers deal only with the retailers, fits readily into this interpretation.

The material of Chapter 6 gives a second limitation on the extent to which the model can be used to explain particular market structures: The results depend on the very strong assumption of rational expectations. However, although the counter-intuitive results presented here would not apply if rational expectations was assumed, the substantive point these results are used to highlight would. That point is that there is an important difference between search-based and non-search-based product-differentiation models; namely, that the differentiation in the former is endogenous.

PART D: CONCLUSION

Chapter 8: Summary And Suggestions For Future Research

8.1 Summary.

Although this thesis has been about equilibrium search, only one of the five main chapters modeled equilibrium directly. Rather than building up specific equilibrium models, the emphasis has been on investigating thoroughly the relationship between the assumptions made about how consumers search and the properties of equilibrium.

The link between search and equilibrium is the demand functions facing firms. Chapter 3 presented a method of deriving demand functions when consumers use FSS search. An important feature of this method is that demand is expressed as a function of the parameters that describe the consumers' sampling strategy. If these demand functions are used in an equilibrium model, the effect on equilibrium of changes in assumptions about search can be easily analysed by adjusting these parameters.

In models of oligopolistic equilibrium, it is usually assumed that demand functions are continuously differentiable. An implication of the approach of Chapter 3 is that, if there are a finite number of firms (but more than two), each firm's demand function will be discontinuous at the other firms prices. This makes it likely that no pure-strategy equilibrium exists with those demand functions. In Chapter 2, it was shown that few search models will produce continuously differentiable demand functions. An exception is the standard model of sequential search, which can be easily made to produce demand functions with the desired property. To do so, however, requires that the search problem is stationary. One condition that is required for stationarity is that sampling be with-replacement. In Chapter 4, it was shown that when there is a finite number of firms, the demand functions will be non-differentiable if without-replacement sequential sampling is assumed.

The implications of non-stationarity were developed further in Chapters 5 and 6 which considered adaptive sequential search. Two main results about equilibrium came from these chapters. The first was that many results of equilibrium-search models only apply if consumers are assumed to have rational expectations about the price distribution. The second was that it can be better to model search in equilibrium using fixed-sample-size search, even in situations where it is optimal for consumers to search sequentially. This is because only simple, non-adaptive models are used in equilibrium; FSS search captures more of the equilibrium properties of adaptive search than does the basic sequential model.

A conclusion of Part B, then, is that non-rational-expectations, without-replacement, FSS search is an appropriate search model to use in equilibrium. The equilibrium model of Chapter 7, however, used rational-expectations, with-replacement, sequential search. The differentiable demand functions that resulted produced robust equilibria which were able to handle the comparative static analysis performed there. These results were interesting, particularly the conclusion that, in a market where there is inefficiency due to imperfect information, the introduction of a specialist dealer in information can lower the efficiency further. The previous chapters, however, show that the assumptions used are very special and crucial to the results. To complete the work of those chapters, ways of incorporating FSS search and non-rational-expectations search into equilibrium with a finite number of firms need to be found. Two ways are suggested in the final section.

8.2 Suggestions for Further research.

The reason that it is difficult to find pure-strategy equilibria in models using FSS search or non-stationary sequential search is because firms are assumed to be be selling a homogeneous product. Discontinuities in the demand functions result from the fact that *all* consumers who have sampled any two firms are indifferent between them except in price. An alternative would to introduce search into a model of product differentiation such as Lancaster (1966). There, atomistic consumers vary continuously in their preferences over similar goods, and so price-setting firms face continuous downward-sloping demand, even when consumers have perfect information. If search was introduced to this model, the more firms that consumers sampled, the closer the substitutes they will find for the product of any particular firm. Search would then affect the elasticity of each firm's demand and the resulting search equilibrium could be compared to the equilibrium without search, rather than to perfect competition.

An alternative to the assumption of rational expectations would be to model reputation with search in a dynamic setting. Consumers would use previous experience to decide *which* firms to sample, and firms would then consider how this period's price would affect next period's search. It may be that such a model would converge to a form of rational expectations in a steady state.

Appendices

APPENDIX A: PROOFS FROM CHAPTER 3

Proof of Lemma 3-1.

We have to show that a necessary and sufficient condition for F_a to stochastically dominate F_b is

$$\sum_{j=1}^{J} p(x_j)\cdot\Big(f_a(x_j) - f_b(x_j)\Big) > 0 \tag{A-1}$$

for all monotonic increasing functions p. By multiplying through by -1, the proof will obviously follow for monotonic decreasing functions when the inequality in (A-1) is reversed.

Sufficient Condition.

F_a fails to stochastically dominate F_b if either

a) $F_a(x_j) = F_a(x_j) \quad \forall\ x_j$, or

b) $F_a(x_j) > F_b(x_j) \quad$ for some x_j.

It is sufficient to show that if either a) or b) are true, then the inequality in (A-1) will not hold for *all* monotonic increasing functions p.

a) If $F_a(x_j) = F_b(x_j) \quad \forall\ j$ then

$$\sum_{j=1}^{J} p(x_j)\cdot\Big(f_a(x_j)-f_b(x_j)\Big) = 0$$

so (A-1) does not hold.

b) Assume there exists k such that $F_a(x_k) > F_b(x_k)$. We need to find one monotonic increasing function p such that (A-1) does not hold.

Let $$p(x_j) = \begin{cases} p_1 & \text{if } j \le k \\ p_2 & \text{if } j > k \end{cases}$$

with $\quad p_1 < p_2$

Then
$$\sum_{j=1}^{J} p(x_j)\cdot\Big(f_a(x_j)-f_b(x_j)\Big) = (p_1-p_2)\cdot\Big(F_a(x_k)-F_b(x_k)\Big) < 0$$

violating (A1).

Necessary Condition.

$$\begin{aligned}\sum_{j=1}^{J} p(x_j)\cdot f_a(x_j) &= p(x_J)\sum_{j=1}^{J} f_a(x_j) \\ &\quad - \Big(p(x_J)-p(x_{J-1})\Big)\sum_{j=1}^{J-1} f_a(x_j) \\ &\quad - \Big(p(x_{J-1})-p(x_{J-2})\Big)\sum_{j=1}^{J-2} f_a(x_j)\ \ldots \\ &= p(x_J)\cdot F_a(x_J) - \sum_{j=1}^{J-1}\Big(p(x_{j+1})-p(x_j)\Big)\cdot F_a(x_j)\end{aligned}$$

so
$$\begin{aligned}\sum_{j=1}^{J} p(x_j)\Big(f_a(x_j)-f_b(x_j)\Big) &= -\sum_{j=1}^{J}\Big(p(x_{j+1})-p(x_j)\Big)\Big(F_a(x_j)-F_b(x_j)\Big) \\ &> 0 \quad \text{if} \quad F_a(x_j) \le F_b(x_j) \quad \forall\ j \\ &\qquad \text{and } p(x_j) \text{ is monotonic increasing.}\end{aligned}$$

This completes the proof for the case where f is discrete. When p and F are defined over the continuous domain $[x_1,x_J]$ the proof of the sufficient condition is essentially unchanged, while the limit form of the necessity proof is

$$\int_{x_1}^{x_J} p(x)\cdot\Big(f_a(x)-f_b(x)\Big)dx = p(x)\cdot\Big(F_a(x)-F_b(x)\Big)\Big|_a^b - \int_{x_1}^{x_J}\frac{dp}{dx}\cdot\Big(F_a(x)-F_b(x)\Big)dx$$

(integrating by parts)

$$= - \int_{x_1}^{x_J} \frac{dp}{dx} \cdot \left(F_a(x) - F_b(x) \right)$$

$$> 0 \qquad \text{if } F_a(x) \le F_b(x) \qquad \forall\ x.$$

□

Proof of Proposition 3-3.

To lighten the notation we write

$$V_m(n) \equiv \sum_{j=1}^{J} V(p_j, p, w-(n-1)c, n) \cdot g_m(p_j)$$

so $V_n(n) \equiv \hat{V}_n$. This notation allows us to separate the benefit of search (increases in m) from the costs (increases in n).

By Assumption 3-2,

$$\frac{\partial^2 V}{\partial n^2} = \left.\frac{\partial^2 V}{\partial n^2}\right|_I + c^2\left(\frac{\partial^2 V}{\partial I^2}\right) -2c\left(\frac{\partial}{\partial I}\left(\left.\frac{\partial V}{\partial n}\right|_I\right)\right) \le 0,$$

so $$V_n(n-1) - V_n(n) \le V_n(n) - V_n(n+1)$$

$$\Rightarrow \quad V_n(n-1) + V_n(n+1) \le 2V_n(n) \tag{A-2}$$

V is a decreasing function in p, so from Assumption 3-1 and Lemma 3-1.

$$V_n(n+1) - V_{n-1}(n+1) > V_{n+1}(n+1) - V_n(n+1) \tag{A-3}$$

Now by Assumption 3-2,

$$\frac{\partial^2 V}{\partial p \partial n} = \frac{\partial}{\partial p}\left(\left.\frac{\partial V}{\partial n}\right|_I\right) - c\frac{\partial^2 V}{\partial p \partial I} \ge 0$$

which implies that $V_n(n-1) - V_n(n+1)$ is a non-increasing function in p, so from Assumption 3-1 and Lemma 3-1,

$$V_n(n-1) - V_n(n+1) \ge V_{n-1}(n-1) - V_{n-1}(n+1)$$

$$\Rightarrow \quad V_n(n-1) - V_{n-1}(n-1) \ge V_n(n+1) - V_{n-1}(n+1) \tag{A-4}$$

Substitution of (A-4) into (A-3) gives

$$V_n(n-1) - V_{n-1}(n-1) > V_{n+1}(n+1) - V_n(n+1)$$

$$\Rightarrow \quad V_n(n-1) + V_n(n+1) > V_{n+1}(n+1) + V_{n-1}(n-1)$$

which, from (A-2), implies

$$V_n(n) - V_{n-1}(n-1) > V_{n+1}(n+1) - V_n(n)$$

$$\Rightarrow \quad \Delta\hat{V}_n > \Delta\hat{V}_{n+1}.$$

Therefore there are diminishing returns to search, with

$$n^* = \min\{\, n \mid n \in \mathbb{Z}_+\backslash 0,\ \Delta\hat{V}_n \le c \,\} \tag{A-5}$$

To show that there exists a monotonic decreasing function C, it is also necessary to show that

$$\frac{\partial\Delta\hat{V}_n}{\partial c} < 0.$$

Now $$\frac{\partial\Delta\hat{V}_n}{\partial c} = (n-1)\cdot\frac{\partial V_n(n)}{\partial I} - n\cdot\frac{\partial V_{n+1}(n+1)}{\partial I}$$

$$< n\cdot\frac{\partial V_n(n)}{\partial I} - n\cdot\frac{\partial V_{n+1}(n+1)}{\partial I}$$

$$< \quad n\cdot\frac{\partial V_{n+1}(n)}{\partial I} - n\cdot\frac{\partial V_{n+1}(n+1)}{\partial I} \tag{A-6}$$

(from Assumptions 3-1 and 3-2 (b), and Lemma 3-1).

Finally, from Assumptions 3-2 (a) and (e),

$$\frac{\partial^2 V}{\partial I \partial n} = \frac{\partial}{\partial I}\left(\left.\frac{\partial V}{\partial n}\right|_I\right) - c\frac{\partial^2 V}{\partial I^2} > 0,$$

so, from (A-6),

$$\frac{\partial \Delta \hat{V}_n}{\partial c} < 0.$$

□

Appendix B: Proofs From Chapter 4

Proof of Proposition 4-3.

The proof is the counterpart to that for Proposition 4-1. We want to show that $Z(s,p_j)$ is non-decreasing in p_j.

a) From (4-2-1),

$$Z(1,p_1) - Z(1,p_2) = \left(S(1,p_2) - S(1,p_1)\right) + \left(V(1,p_2) - V(1,p_1)\right).$$

Now $\quad S(1,p_1) \geq S(1,p_2)$

(from Assumption 4-1 (b)),

and $\quad V(1,p_1) \geq V(1,p_2)$

so $\quad Z(1,p_1) \leq Z(1,p_2)$

This gives the induction base.

b) Assume that $Z(s-1,p_j)$ is non-decreasing in p_j.

Then $$U(s-1,p_{j+1}) - V(s-1,p_{j+1}) \geq U(s-1,p_j) - V(s-1,p_j) \tag{B-1}$$

From (4-2-1),

$$\begin{aligned} Z(s,p_{j+1}) - Z(s,p_j) = {} & \left(\frac{s-j}{s}U(s-1,p_{j+1}) + \frac{1}{s}\sum_{i=1}^{j-1} U(s-1,p_i) + \frac{1}{s}U(s-1,p_j) \right. \\ & \left. - S(s,p_{j+1}) - V(s,p_{j+1})\right) \\ & - \left(\frac{s-j+1}{s}U(s-1,p_j) + \frac{1}{s}\sum_{i=1}^{j-1} U(s-1,p_i) - S(s,p_j) - V(s,p_j)\right) \end{aligned}$$

$$= \frac{s-j}{s}\left(U(s-1,p_{j+1})-U(s-1,p_j)\right) + \left(V(s,p_j)-V(s,p_{j+1})\right) + \left(S(s,p_j)-S(s,p_{j+1})\right)$$

$$\geq \frac{s-j}{s}\left(U(s-1,p_{j+1})-U(s-1,p_j) + V(s,p_j)-V(s,p_{j+1})\right) + \left(S(s,p_j)-S(s,p_{j+1})\right)$$

$$\geq \frac{s-j}{s}\Big(U(s-1,p_{j+1})-U(s-1,p_j) + V(s-1,p_j)-V(s-1,p_{j+1})\Big) + \Big(S(s,p_j)-S(s,p_{j+1})\Big)$$

(from Assumption 4-1 a)).

So, from equation (B-1) and Assumption 4-1 (b),

$$Z(s,p_{j+1}) - Z(s,p_j) \geq 0.$$

This completes the induction step: The proposition holds for s-1, so it holds for s.

□

Proof of Proposition 4-4.

Following the procedure of the proof of Proposition 4-2, let p_k be the reservation price at (s-1) so $Z(s-1,p_k) \leq 0$. The proposition will be proved by showing that $Z(s,p_k) \leq 0$.

Now $$Z(s,p_k) = \frac{s-k+1}{s}V(s-1,p_k) + \frac{1}{s}\sum_{i=1}^{k-1} V(s-1,p_i) - S(s,p_k) - V(s,p_k)$$

and $$Z(s-1,p_k) = \frac{s-k+1}{s}U(s-2,p_k) + \frac{1}{s}\sum_{i=1}^{k-1} U(s-2,p_i)$$
$$- \frac{1}{s(s-1)}\sum_{i=1}^{k-1}\Big(U(s-2,p_k)-U(s-2,p_i)\Big) - S(s-1,p_k) - V(s-1,p_k)$$

so,

$$Z(s-1,p_k) - Z(s,p_k) = \frac{s-k+1}{s}\Big(U(s-2,p_k)-V(s-1,p_k)\Big) + \frac{1}{s}\sum_{i=1}^{k-1}\Big(U(s-2,p_i)-V(s-1,p_i)\Big)$$
$$- \frac{1}{s(s-1)}\sum_{i=1}^{k-1}\Big(U(s-2,p_k)-U(s-2,p_i)\Big)$$
$$+ \Big(V(s,p_k)-V(s-1,p_k)\Big) + \Big(S(s,p_k)-S(s-1,p_k)\Big)$$

Now, $U(s-2,p_k) - U(s-2,p_i) \le 0$

and $S(s,p_k) - S(s-1,p_k) \ge 0$

(from Assumption 4-2 (b)),

and

$$\frac{s-k+1}{s}\Big(U(s-2,p_k-V(s-1,p_k)\Big) + \frac{1}{s}\sum_{i=1}^{k-1}\Big(U(s-2,p_i)-V(s-1,p_i)\Big) + \Big(V(s,p_k)-V(s-1,p_k)\Big)$$

$$\ge \frac{s-k+1}{s}\Big(V(s-2,p_k)-V(s-1,p_k)\Big) + \frac{1}{s}\sum_{i=1}^{k-1}\Big(V(s-2,p_i)-V(s-1,p_i)\Big) + \Big(V(s,p_k)-V(s-1,p_k)\Big)$$

(by the definition of $U(s-2,p_j)$)

$$\ge \frac{s-k+1}{s}\Big(V(s-2,p_k)-V(s-1,p_k)\Big) + \frac{1}{s}\sum_{i=1}^{k-1}\Big(V(s-2,p_k)-V(s-1,p_k)\Big) + \Big(V(s,p_k)-V(s-1,p_k)\Big)$$

(from Assumption 4-2 (c))

$$= \Big(V(s,p_k)-V(s-1,p_k)\Big) - \Big(V(s-1,p_k)-V(s-2,p_k)\Big)$$

which is positive from Assumption 4-2 (a).

□

Derivation of $C(s,p_j)$.

The following lemma will be used in the derivation.

Lemma B-1:

$$\sum_{k=1}^{s-1}\frac{(k+j)!}{(k-1)!} = \frac{(s+j)!}{(s-2)!\,(j+2)} \qquad \forall\ j,\ \forall\ s\ge 2$$

Proof:

The proof is by induction. Clearly the lemma holds for s=2. Assume that it holds for s, then

$$\sum_{k=1}^{s-1}\frac{(k+j)!}{(k-1)!} = \frac{(s+j)!}{(s-2)!\,(j+2)}$$

$$\begin{aligned}\Rightarrow \quad \sum_{k=1}^{s}\frac{(k+j)!}{(k-1)!} &= \frac{(s+j)!}{(s-2)!\,(j+2)} + \frac{(s+j)!}{(s-1)!} \\ &= \frac{(s+j)!\,(s-1)}{(s-1)!\,(j+2)} + \frac{(s+j)!\,(j+2)}{(s-1)!\,(j+2)} \\ &= \frac{(s+j+1)}{(s-1)!\,(j+2)}\end{aligned}$$

□

From equations (4-3-1) and (4-3-2),

$$A(s,p_j) = \min\left\{ p_j, \frac{s-j+1}{s}A(s-1,p_j) + \frac{1}{s}\sum_{i=1}^{j-1} A(s-1,p_i) + c \right\} \tag{B-2}$$

If $Z(s,p_j) < 0$ so that search continues at (s,p_j), then Proposition 4-2 implies that it continues at all (t,p_j) $t > s$. The second argument of (B-2) can therefore be expanded out until there are no $A(t,p_j)$ terms left:

$$\begin{aligned}&\frac{s-j+1}{s}A(s-1,p_j) + \frac{1}{s}\sum_{i=1}^{j-1} A(s-1,p_i) + c \\ &= \left(1 + \frac{s-j+1}{s} + \left(\frac{s-j+1}{s}\right)\left(\frac{s-j+2}{s-1}\right) + \cdots\right)c \\ &\qquad + \left(\frac{1}{s}\sum_{i=1}^{j-1} A(s-1,p_i) + \left(\frac{s-j+1}{s-1}\right)\left(\frac{1}{s-2}\right)\sum_{i=1}^{j-1} A(s-2,p_i) \cdots\right) \\ &= \sum_{k=1}^{s-j+2}\left(\left(\frac{(s-j+1)!/(k-1)!}{s!/(k+j-2)!}\right)c + \frac{(s-j+1)!/(k-1)!}{s!/(k+j-3)!}\sum_{i=1}^{j-1} A(k+j-3,p_i)\right) \\ &= \frac{(s-j+1)!}{s!}\left(\sum_{k=1}^{s-j+2}\left(\left(\frac{(k+j-2)!}{(k-1)!}\right)c + \frac{(k+j-3)!}{(k-1)!}\sum_{i=1}^{j-1} A(k+j-3,p_i)\right)\right)\end{aligned}$$

$$= \frac{1}{j}(s+1)c + \frac{(s-j+1)!}{s!}\sum_{k=1}^{s-j+2}\frac{(k+j-3)!}{(k-1)!}\sum_{i=1}^{j-1} A(k+j-3,p_i) \tag{B-3}$$

(using Lemma B-1.)

Equation (B-2) can then be re-expressed

$$A(s,p_j) = \frac{(s-j)!}{s!}\min\left\{\left(\frac{s!}{(s-j)!}\right)p_j,\ \left(\frac{(s+1)!}{(s-j)!\,j}\right)c \right.$$
$$\left. + (s-j+1)\sum_{i=1}^{j-1}\sum_{k=1}^{s-j+2}\frac{(k+j-3)!}{(k-1)!}A(k+j-3,p_i)\right\} \tag{B-4}$$

Taking the last part of equation (B-4),

$$\sum_{h=1}^{j-1}\sum_{k=1}^{s-j+2}\frac{(k+j-3)!}{(k-1)!}A(k+j-3,p_i)$$
$$= \sum_{h=1}^{j-2}\sum_{k=1}^{s-j+2}\frac{(k+j-3)!}{(k-1)!}A(k+j-3,p_i) + \sum_{k=1}^{s-j+2}\frac{(k+j-3)!}{(k-1)!}A(k+j-3,p_{j-1}) \tag{B-5}$$

Let x_j be the image reservation price for p_j. That is, let x_j be the lowest s such that search terminates at (s,p_j). Then,

$$A(k+j-3,p_{j-1}) = p_{j-1} \quad \forall \quad k > x_{j-1}-j+2$$

and for all $k \le x_{j-1}-j+2$, $A(k+j-3,p_{j-1})$ can be expanded out as in (B-3) substituting $(k+j-3)$ for s, and $(j-1)$ for j. Therefore the second term of equation (B-5) can be re-expressed,

$$\sum_{k=1}^{s-j+2}\frac{(k+j-3)!}{(k-1)!}A(k+j-3,p_{j-1}) = \sum_{k=x_{j-1}-j+3}^{s-j+2}\left(\frac{(k+j-3)!}{(k-1)!}\right)p_{j-1} + \left(\sum_{k=1}^{x_{j-1}-j+2}\frac{(k+j-3)!}{(k-1)!}\right.$$
$$\left.\times\left(\left(\frac{k+j-2}{j-1}\right)c + \frac{(k-1)!}{(k+j-3)!}\left(\sum_{h=1}^{k}\frac{(h+j-4)!}{(h-1)!}\sum_{i=1}^{j-2}A(h+j-4,p_i)\right)\right)\right)$$

$$= \sum_{k=x_{j-1}-j+3}^{s-j+2} \left(\frac{(k+j-3)!}{(k-1)!}\right) p_{j-1} + \sum_{k=1}^{x_{j-1}-j+2} \left(\frac{(k+j-2)!}{(k-1)!\,(j-1)}\right) c$$

$$+ \sum_{i=1}^{j-2} \left(\sum_{k=1}^{x_{j-1}-j+3} \left(x_{j-1}-j-k+3\right) \frac{(k+j-4)!}{(k-1)!} A(k+j-4, p_i) \right) \qquad \text{(B-6)}$$

The first term of equation (B-5) can be written,

$$\sum_{i=1}^{j-2} \left(\sum_{k=1}^{s-j+2} \frac{(k+j-3)!}{(k-1)!} A(k+j-3, p_i) \right) = \sum_{i=1}^{j-2} \left(\sum_{k=1}^{x_{j-1}-j+3} \frac{(k+j-4)!\,(k-1)}{(k-1)!} A(k+j-4, p_i) \right)$$

$$+ \sum_{i=1}^{j-2} \left(\sum_{k=x_{j-1}-j+3}^{s-j+2} \frac{(k+j-3)!}{(k-1)!} A(k+j-3, p_i) \right) \qquad \text{(B-7)}$$

Proposition 4-1 and the definition of x_j imply that

$$A(t, p_i) = p_i \qquad \forall \quad t \geq x_j, \; i \leq j.$$

so that

$$A(k+j-3, p_i) = p_i \qquad \forall \quad k \geq x_{j-1}-j+3, \; i \leq j-1$$

so p_i can be substituted for $A(k+j-3, p_i)$ in the last term of (B-7).

Adding (B-7) to (B-6), (B-5) can be reexpressed,

$$\sum_{i=1}^{j-1} \left(\sum_{k=1}^{s-j+2} \frac{(k+j-3)!}{(k-1)!} A(k+j-3, p_i) \right)$$

$$= \left(\sum_{k=1}^{x_{j-1}-j+2} \left(\frac{(k+j-2)!}{(k-1)!\,(j-1)}\right) c + \sum_{i=1}^{j-1} \left(\sum_{k=x_{j-1}-j+3}^{s-j+2} \left(\frac{(k+j-3)!}{(k-1)!}\right) p_i \right) \right)$$

$$+ \left(x_{j-1}-j+2\right) \sum_{i=1}^{j-2} \left(\sum_{k=1}^{x_{j-1}-j+3} \frac{(k+j-4)!}{(k-1)!} A(k+j-4, p_i) \right)$$

Now

$$\sum_{k=x_{j-1}-j+3}^{s-j+2}\left(\cdot\right) = \sum_{k=1}^{s-j+2}\left(\cdot\right) - \sum_{k=1}^{x_{j-1}-j+2}\left(\cdot\right)$$

so using Lemma B-1 to reexpress the coefficients on c and p_i

$$\begin{aligned}\left(s-j+1\right)\sum_{k=1}^{s-j+2}\frac{(k+j-3)!}{(k-1)!}A(k+j-3,p_i) &= (s-j+1)\Bigg[\left(\frac{(x_{j-1}+1)!\,(x_{j-1}-j+2)}{(x_{j-1}-j+2)!\,j(j-1)}\right)c \\ &\quad + \sum_{i-1}^{j-1}\left(\frac{s!}{(s-j+1)!\,(j-1)} - \frac{x_{j-1}!\,(x_{j-1}-j+2)}{(x_{j-1}-j+2)!\,(j-1)}\right) \\ &\quad + \left(x_{j-1}-j+2\right)\sum_{i=1}^{j-2}\left(\sum_{k=1}^{x_{j-1}-j+3}\frac{(k+j-4)!}{(k-1)!}A(k+j-4,p_i)\right)\Bigg]\end{aligned} \qquad \text{(B-8)}$$

The numerator and denominator are multiplied by $x_{j-1}-j+2$ in the coefficients for c and p_i to keep the expression defined when $x_{j-1} = j-2$.

Equation (B-8) is the last term of (B-4) expanded out to eliminate the $A(s-1,p_i)$ term. The last term of (B-8) can be iteratively expanded in a similar way (extending the coefficient $(s-j+1)\cdot(x_{j-1}-j+2)\cdots$ with each iteration) so that, from (B-4),

$$\begin{aligned}A(s,p_j) = \frac{(s-j)!}{s!}\cdot\min\Bigg\{&\left(\frac{s!}{(s-j)!}\right)p_j,\ \left(\frac{(s+1)!}{(s-j)!\,j}\right)c + \sum_{i=1}^{j-1}\left(\prod_{k=1}^{j-1}(x_{k+1}-k)\right) \\ &\times\left(\left(\frac{(x_i+1)!\,(x_i-i+1)}{(x_{i-i}+1)!\,i(i+1)}\right)c + \sum_{h=1}^{i}\left(\frac{x_{i+1}!}{(x_{i+1}-i)!\,i} - \frac{x_i!\,(x_i-i+1)}{(x_{i-i}+1)!\,i}\right)p_h\right)\Bigg\}\end{aligned} \qquad \text{(B-9)}$$

where by definition

$$x_j \equiv s,\ x_1 \equiv 0; \qquad \text{and} \quad i-1 \le x_i \le x_{i+1} \quad \forall \quad 1 < i < j.$$

Now, $C_{\mathbf{x}}(s,p_j)$ is defined so that the two terms in the min function of (B-9) are equal. Collecting terms and rearranging gives

$$C(s,p_j) = \frac{N(p_j)}{D(p_j)} \qquad j > 1$$

with $N(p_j)$ and $D(p_j)$ as defined in equations (4-4-5) and (4-4-6)

□

Proof of Proposition 4-5.

If $p_j = p_{j+1}$, then $x_j = x_{j+1} = s$ in the definition of $C(s,p_{j+1})$. Therefore, from (4-4-6),

$$D(p_{j+1}) = \frac{(s+1)!}{(s-j-1)!(j+1)} + \sum_{i=1}^{j}\left(\prod_{k=i}^{j}(x_{k+1}-k)\right)\frac{(x_i+1)!(x_i-i+1)}{(x_{i-i}+1)!\,i(i+1)}$$

$$= \frac{(s+1)!}{(s-j-1)!\,j} - \frac{(s+1)!}{(s-j-1)!\,j(j+1)} + \left(s-j\right)\sum_{i=1}^{j-1}\left(\prod_{k=i}^{j-1}(x_{k+1}-k)\right)\frac{(x_i+1)!(x_i-i+1)}{(x_{i-i}+1)!\,i(i+1)} + \left(s-j\right)\frac{(s+1)!(s-j+1)}{(s-j+1)!\,j(j+1)}$$

$$= (s-j)\frac{(s+1)!}{(s-j)!\,j} - \frac{(s+1)!}{(s-j-1)!\,j(j+1)} + \left(s-j\right)\sum_{i=1}^{j-1}\left(\prod_{k=i}^{j-1}(x_{k+1}-k)\right)\frac{(x_i+1)!(x_i-i+1)}{(x_{i-i}+1)!\,i(i+1)} + \frac{(s+1)!}{(s-j-1)!\,j(j+1)}$$

$$= (s-j)D(p_j)$$

and, from (4-4-5),

$$N(p_{j+1}) = \frac{s!}{(s-j-1)!}p_j + (s-j)\sum_{i=1}^{j-1}\left(\prod_{k=i}^{j-1}(x_{k+1}-k)\right)\sum_{h=1}^{i}\left(\frac{x_i!\,(x_i-i+1)}{(x_{i-i}+1)!\,i} - \frac{x_{i+1}!}{(x_{i+1}-i)!\,i}\right)p_h$$

$$= (s-j)\cdot N(p_j)$$

so $$C(s,p_{j+1}) = \frac{(s-j)N(p_j)}{(s-j)D(p_j)} = C(s,p_j)$$

$\square$

Appendix C: Proofs From Chapter 5

Derivation of $\hat{p}(x_n|\hat{x})$.

Let $\hat{p}_{n\ell}(p)$ be the density function of the minimum price sampled after n quotations when $\hat{x}$ is known and let $\hat{p}_{nh}(p|p_{n\ell})$ be the density function of the maximum price conditional on both $\hat{x}$ and the sample minimum $p_{n\ell}$. Then

$$\hat{p}(x_n|\hat{x}) = \int \hat{p}_{nh}(p_{n\ell}+x_n|p_{n\ell})\cdot\hat{p}_{n\ell}(p)dp \qquad \text{(C-1)}$$

In a sample of size n, the probability that any given price exceeds some value p′ is

$$\frac{a+\hat{x}-p'}{\hat{x}} \qquad a < p' < a + \hat{x}$$

where a is the infimum of the price support. The probability that all n prices do is

$$\left(\frac{a+\hat{x}-p'}{\hat{x}}\right)^n \qquad a < p' < a + \hat{x}$$

So $$\Pr(p_{n\ell}\le p') = 1 - \left(\frac{a+\hat{x}-p'}{\hat{x}}\right)^n$$

giving $$\hat{p}_{n\ell}(p) = \frac{n}{\hat{x}}\left(\frac{a+\hat{x}-p}{\hat{x}}\right)^{n-1} \qquad \text{(C-2)}$$

Given $p_{n\ell}$, the probability that all the remaining (n−1) prices do not exceed some value p′ is

$$\left(\frac{p'-p_{n\ell}}{a+\hat{x}-p_{n\ell}}\right)^{n-1}$$

giving $$\hat{p}_{nh}(p'|p_{n\ell}) = \frac{n-1}{a+\hat{x}-p_{n\ell}}\left(\frac{p'-p_{n\ell}}{a+\hat{x}-p_{n\ell}}\right)^{n-2} \tag{C-3}$$

Equations (C-2) and (C-3) substituted into (C-1) give

$$\hat{p}(x_n|\hat{x}) = \int_a^{a+\hat{x}-x_n} n(n-1)\frac{x_n^{n-2}}{\hat{x}^n}\,dp$$

$$= n(n-1)\frac{x_n^{n-2}}{\hat{x}^{n-1}}\left(1-\frac{x_n}{\hat{x}}\right)$$

Proof of Proposition 5-1.

a). $G_n(0)$ follows directly from (5-4-12); $\lim_{x \to X} G_n(x) = 0$ is established by applying L'H pital's rule twice.

b). First, note from equation (5-4-12) that $G_n(x_n) > 0 \;\; \forall\, x \in (0,X)$. For the proposition to be false, it is necessary that there exist x_n^1, x_n^2, x_n^3, where $x_n^1 < x_n^2 < x_n^3$, such that

$$G_n'(x_n^i) = 0 \qquad \text{for } i \in \{1,2,3\};$$

and $$G_n''(x_n^1) < 0, \quad G_n''(x_n^2) > 0, \quad G_n''(x_n^3) < 0.$$

Let f(R) and g(R) be the numerator and denominator of (5-4-12). From the quotient rule for differentiation,

$$(f/g)''\big|_{((f/g)'=0)} = \frac{f''\cdot g' - f'\cdot g''}{g\cdot g'}.$$

Let $$A(R) = \frac{(f''\cdot g' - f'\cdot g'')}{6XR^{n-4}}$$

$$= \int_R^1 (1-\hat{R})^3\hat{R}^{n-4}d\hat{R}\Big((n-3)-(n-2)R\Big) + R^{n-3}\Big(1-6R+12R^2-10R^3+3R^4\Big)$$

Since $g' < 0$, it is sufficient to show that there exists R^A such that

$$A(R^A) > 0 \qquad \text{for } R\in(0,R^A)$$

and $$A(R^A) > 0 \qquad \text{for } R\in(R^A,1).$$

It is easily demonstrated that

i) $A(0) > 0, \quad A(1) = 0;$

ii) $A'(0) < 0, \quad A'(1) = 0;$

iii) $A''(R) = R^{n-4}B(R),$

iv) $B(1) = 0$ and $B'(1) < 0.$

where $$B(R) = -(2n^2-8n+5) + 6n(n-2)R - 3(2n^2-1)R^2 + 2(n+1)^2R^3$$

Since $B(R)$ is a positive cubic, iv), iii), and ii) imply that there exists R^B such that

$$A'(R^B) < 0 \quad \forall\ R \in (0,R^B) \qquad \text{and} \qquad A'(R^B) > 0 \quad \forall\ R \in (R^B,1).$$

With i), this implies the existence of R^A.

c). Note that

$$\frac{\partial G_n(x_n)}{\partial x_n} < \frac{G_n(x_n)}{x_n} \quad \text{iff} \quad \frac{\partial G_n(XR)}{\partial R} < \frac{G_n(XR)}{R}.$$

Let $$C(R) = \frac{G_n(RX)}{R}$$

We want to show that

$$\frac{\partial}{\partial R}\Big(RC(R)\Big) < C(R)$$

$$\Rightarrow \quad C'(R) < 0.$$

Using the quotient rule,

$$C'(R) = \frac{6XR^{n-4}(1-R)}{g(R)^2}D(R)$$

where $$D(R) = R\int_R^1 (1-\hat{R})^3\hat{R}^{n-4}d\hat{R} - (1-R)^2\int_R^1 \hat{R}^{n-3}(1-\hat{R})d\hat{R}$$

Now, $D(0) < 0$, $D(1) = 0$, and

$$D'(R) = \int_R^1 (1-\hat{R})^3\hat{R}^{n-4}d\hat{R} + 2(1-R)\int_R^1 \hat{R}^{n-3}(1-\hat{R})d\hat{R} > 0,$$

so $D(R) < 0$, and hence $C'(R) < 0 \;\; \forall \;\; R \in (0,1)$.

d). We want to show that

$$\frac{\int_R^1 (1-\hat{R})^3\hat{R}^{n-4}d\hat{R}}{\int_R^1 (1-\hat{R})\hat{R}^{n-3}d\hat{R}} - \frac{\int_R^1 (1-\hat{R})^3\hat{R}^{n-3}d\hat{R}}{\int_R^1 (1-\hat{R})\hat{R}^{n-2}d\hat{R}} > 0$$

$$\Rightarrow \quad \frac{\int_R^1 (1-\hat{R})\hat{R}^{n-2}d\hat{R}}{\int^1 (1-\hat{R})\hat{R}^{n-3}d\hat{R}} - \frac{\int_R^1 (1-\hat{R})^3\hat{R}^{n-3}d\hat{R}}{\int^1 (1-\hat{R})^3\hat{R}^{n-4}d\hat{R}} > 0 \qquad \text{(C-4)}$$

We can write the LHS of (C-4) as

$$\int_R^1 \hat{R}\cdot q_1(\hat{R})d\hat{R} - \int_R^1 \hat{R}\cdot q_2(\hat{R})d\hat{R}$$

where $q_1(\hat{R}) = \dfrac{(1-\hat{R})\hat{R}^{n-3}d\hat{R}}{\int_R^1 (1-R')R'^{n-3}dR'}$ and $q_2(\hat{R}) = \dfrac{(1-\hat{R})^3\hat{R}^{n-4}d\hat{R}}{\int_R^1 (1-R')^3R'^{n-4}dR'}$

can both be interpreted as probability density functions. The inequality of (C-4) will then hold if q_2 stochastically dominates q_1. From Lemma 3-1, p 36, this implies

$$H(\lambda) \equiv \int_R^{\lambda} \left(q_1(\hat{R}) - q_2(\hat{R})\right) d\hat{R} \ < 0 \qquad \forall \ \lambda \in (R,1).$$

Note that

$$H(R) = H(1) = 0 \tag{C-5}$$

and $$H'(\lambda) = (1-\lambda)\lambda^{n-3}\left(\frac{\lambda}{\int (1-R')R'^{\,n-3}dR'} - \frac{(1-\lambda)^2}{\int (1-R')R'^{\,n-4}dR'} \right)$$

The bracketed term is a concave function of λ so, from (C-5),

$$H(\lambda) < 0 \qquad \forall \ \lambda \in (R,1)$$

□

Proof of Proposition 5-3.

The proposition is proved with the aid of a series of lemmas.

Lemma C-1:

a) $$\frac{\partial p_n(x|x_n)}{\partial x_n} > 0 \qquad \forall \ x \in [x_n, X]$$

b) $$\frac{\partial p_n(x_n|x_n)}{\partial x_n} > 0.$$

Proof:

Let $\hat{p}(x|x_n)$ be the probability distribution of x_{n+1} conditional on x_n and $\hat{x}$. So

$$p_n(x|x_n) = \int \hat{p}_n(x|x_n)\cdot p(\hat{x}|x_n)d\hat{x} \tag{C-6}$$

Now

$$x_{n+1} = \begin{cases} x_n + p_{n\ell} - p & p < p_{n\ell} \\ x_n & p_{n\ell} \le p \le p_{nh} \\ x_n + p - p_{nh} & p_{nh} < p \end{cases}$$

$\hat{p}(x|x_n)$ is then a mixed distribution, calculated from (5-4-9),

$$\hat{p}(x|x_n) = \begin{cases} \dfrac{2(\hat{x}-x)}{\hat{x}(\hat{x}-x_n)} & \hat{x} > x > x_n \quad \text{(a density function)} \\ \dfrac{x_n}{\hat{x}} & \hat{x} > x = x_n \quad \text{(a probability mass)} \end{cases} \tag{C-7}$$

Let $R = \frac{x}{X}$ and $R_n = \frac{x_n}{X}$. Substituting (C-7) and (5-4-10) into (C-6) gives

$$p_n(x|x_n) = \frac{2\displaystyle\int_{R_n}^{R_n/R} \left(1-\frac{R}{R_n}\hat{R}\right)\hat{R}^{n-2}d\hat{R}}{R_n X\displaystyle\int_{R_n}^{1} (1-R')R'^{\,n-3}dR'}$$

$$= \frac{\left(\frac{2}{X}\right)R_n^{n-2}\left(1-nR^{n-1}+(n-1)R^{n}\right)}{n\;n\;1)R^{n-1}\displaystyle\int_{R_n}^{1} (1-R')R'^{\,n-3}dR'} \qquad \text{for} \quad x < x_n \tag{C-8}$$

and
$$p_n(x_n|x_n) = \frac{\displaystyle\int_{R_n}^{1} (1-\hat{R})\hat{R}^{n-2}d\hat{R}}{\displaystyle\int_{R_n}^{1} (1-R')R'^{\,n-3}dR'} \tag{C-9}$$

The numerator of equation (C-8) increases in R_n; the denominator decreases. Part a) of the proposition follows since $\partial R_n/\partial x_n > 0$.

To show b), from (C-9),

$$\frac{\partial p_n(x_n|x_n)}{\partial x} = \frac{(1-R_n)R_n^{n-3}\left(\displaystyle\int_{R_n}^{1} (1-\hat{R})\hat{R}^{n-2}d\hat{R} - R_n\int_{R_n}^{1} (1-\hat{R})\hat{R}^{n-3}d\hat{R}\right)}{\left(\displaystyle\int_{R_n}^{1} (1-\hat{R})\hat{R}^{n-3}d\hat{R}\right)^2} > 0$$

□

Lemma C-2:

$$G_n^*(x_n) \geq G_n(x_n) \qquad \forall\ n, \quad \forall\ x_n \in (0,X)$$

Proof:

Follows from equation (5-4-14) and the definition of $\mathbb{X}_n$.

□

Lemma C-3:

$$G_n^*(x_n) = G_n(x_n) \qquad \forall\ x_n \in [x_{n\hbar}, X].$$

Proof:

We want to show that

$$\int_{\mathbb{X}_n} (G_{n+1}^*(x)-c)\cdot p(x|x_n)dx = 0 \quad \text{for } x_n \in [x_{n\hbar}, X).$$

Note that $p(x|x_n)=0$ for $x<x_n$ so we only need to show that $x \notin \mathbb{X}_n$ for any $x \in [x_{n\hbar}, X]$. The proof is by induction.

a) By definition,

$$G_{J-1}^*(x_n) = G_{J-1}(x_n) \tag{C-10}$$

(as only one firm remains to be searched)

b) Assume that

$$G_{n+1}^*(x_n) = G_{n+1}(x_n) \quad \forall\ x_{n+1} \in [x_{(n+1)h}, X] \tag{C-11}$$

From Proposition 5-1,

$$G_{n+1}(x_n) < G_n(x_n) \qquad \forall\ x_{n+1} \geq x_n;$$

so if (C-11) holds,

$$x_n \in [x_{nh}, X] \Rightarrow G_{n+1}(x_n)-c < 0$$

$$\Rightarrow G^*_{n+1}(x_{n+1})-c < 0 \qquad \forall\ x_{n+1} \geq x_n,$$

and so $\quad x_{n+1} \notin \mathbb{X}_n.$

□

Lemma C-4:

For all $x_n \geq x_{n\ell}$, the decision to continue search is independent of the opportunity for future search. Formally,

$$G^*_n(x_n) \lesseqgtr c \quad \text{as} \quad G_n(x_n) \lesseqgtr c \qquad \forall\ x_n \in [x_{n\ell}, X].$$

Proof:

The lemma follows for $x_n \in (x_{n\ell}, x_{nh})$ from Lemma C-2, and for $x_n \in [x_{nh}, X]$ from Lemma C-3.

□

Lemma C-4 implies that x^*_{nh} exists and is equal to x_{nh}. Proposition 5-3 b) then follows from Proposition 5-2 b) for the upper limit. To complete the proof of a), it is sufficient to show that $G^*_n(0) = 0$, and that, for $x \in [0, x_{n\ell}]$, $G^*_n(x_n)$ is an increasing function. Again, the proof is by induction,

a) From (C-10) and Proposition 5-2, Proposition 5-3 a) holds for n=J-1.

b) Assume that it holds for n+1. The set $\mathbb{X}_n$ is then a compact interval, and we can rewrite (5-4-14)

$$G^*_n(x_n) = G_n(x_n) + \int_{x^*_{(n+1)\ell}}^{x^*_{(n+1)h}} (G^*_{n+1}(x)-c)\cdot p(x|x_n)\ dx \tag{C-12}$$

$G^*_n(0)=0$, since, from Proposition 5-1 a), $G_n(0)=0$, and from (C-11) $p_n(x|0)=0$. Let $x'_{n\ell}$ be the unique value of x_n such that

i) $G^*_n(x'_{n\ell}) = c$; and

ii) $G^*_n(x_n) > c \qquad \forall\ x_n \in (x'_{n\ell}, x'_{nh})$.

That is, $x'_{n\ell}$ is the highest value of x_n in the range $(0, x_{nh})$ such that i) holds. We want to show that it is the only value. From Proposition 5-1, Lemma C-1, and the induction base, $G^*_n(x_n)$ is increasing in x_n for $x_n < x'_{n\ell}$. It follows that $x^*_{n\ell}$ exists and is equal to $x'_{n\ell}$.

Finally, note that

$$G_n(x_n) > G_{n+1}(x_n) \qquad \text{from Proposition 5-1 b); and}$$

$$G^*_{n+1}(x_n) - c > G^*_{n+2}(x_n) \qquad \text{from the induction base.}$$

Unfortunately, it is not always the case that $p_n(x|x_n) > p_{n+1}(x|x_n)\ \forall\ x > x_n$, so the natural proof that $x^*_{n\ell}$ converges monotonically to x^*_{nh} does not follow. To show that $x^*_{n\ell}$ does converge if $x^*_{(J-1)\ell} < x^*_{(J-1)h}$ note from (C-12) that

$$x^*_{n\ell} \le x_{n\ell}\ ;$$

from (C-10),

$$x^*_{(J-1)\ell} = x_{(J-1)\ell}\ ;$$

and from Proposition 5-2 b),

$$x_{(J-1)\ell} > x_{n\ell} \qquad \text{for} \quad 1 < n < J-1.$$

It follows that

$$x^*_{(J-1)\ell} > x^*_{n\ell} \qquad \text{for} \quad 1 < n < J-1.$$

□

Proof of Theorem 5-2.

a). $\frac{\partial x^*_{n\ell}}{\partial c} > 0$ and $\frac{\partial x^*_{nh}}{\partial c} < 0$ follow from the fact that $\frac{\partial}{\partial x}G^*_n(x^*_{n\ell}) > 0$ and $\frac{\partial}{\partial x}G^*_n(x^*_{nh}) < 0$ as shown by the proof to Proposition 5-3. This implies (from equation (C-12)) that $\frac{\partial}{\partial c}G^*_n(x_n) < 0 \quad \forall\ x_n \in (0, x^*_{(n+1)h})$, strengthening the result.

b). Let $\Delta_J G^*_n(x_n)$ be the increase in $G^*_n(x_n)$ as a result of increasing the number of firms from J to J+1. From (C-10) and (C-12),

$$G^*_n(x_n) = \begin{cases} G_n(x_n) & n = J-1 \\ G_n(x_n) + \displaystyle\int_{x^*_{(n+1)\ell}}^{x^*_{(n+1)h}} (G^*_{n+1}(x)-c)\cdot p(x|x_n)dx & n < J-1 \end{cases}$$

so
$$\Delta_{n+1}G^*_n(x_n) > 0 \quad \text{if} \quad \int_{x^*_{(n+1)\ell}}^{x^*_{(n+1)h}} (G^*_{n+1}(x)-c)\cdot p(x|x_n)dx > 0 \qquad \text{(C-13)}$$
$$= 0 \quad \text{otherwise;}$$

and, iterating backwards,

$$\Delta_J G^*_{n-1}(x_n) > 0 \quad \text{iff} \quad \Delta_J G^*_n(x_n) > 0 \qquad \forall\ n < J-1.$$

It follows from (C-13) and Theorem 5-1 that $\Delta_J G^*(x_{n\ell}) > 0$ iff $J < J^* = n^*$.

Finally, since $G^*_n(x_{nh}) = G_n(x_{nh})$, $\Delta_J G^*_n(x_{nh}) = 0$.

c). To show that search intensity increases with X, it is sufficient to show that $\frac{\partial}{\partial X}G^*_n(x_n) > 0 \quad \forall\ x_n \in (0, X]$. First consider the effect of X on $G_n(x_n)$

Lemma C-5:

$$\frac{\partial}{\partial X} G_n(x_n) > 0 \qquad \forall\ x \in (0, X).$$

Proof:

Define x^K so that $G_n(x^K) \equiv K$. We want to show that $\frac{\partial x^K}{\partial X}$ is negative when $G_n(x_n)$ is increasing and positive when $G_n(x_n)$ is decreasing. Let

$$R^K \equiv \frac{x^K}{X}.$$

Then,

$$\frac{\partial x^K}{\partial X} = R^K + X\frac{\partial R^K}{\partial X}. \tag{C-14}$$

By the implicit function theorem,

$$\frac{\partial R^K}{\partial X} = - \frac{\left.\frac{\partial G(XR^K)}{\partial X}\right|_R}{\left.\frac{\partial G(XR^K)}{\partial R}\right|_X} \tag{C-15}$$

Now, from (5-4-12),

$$X\left.\frac{\partial G(XR^K)}{\partial X}\right|_R = G(XR^K)$$

so (C-15) into (C-14) gives,

$$\frac{\partial x^K}{\partial X} = R^K - \frac{G(XR^K)}{\left.\frac{\partial G(XR^K)}{\partial R}\right|_X}$$

and so
$$\operatorname{sgn}\left\{\frac{\partial x^K}{\partial X}\right\} = \begin{cases} \operatorname{sgn}\left\{\left.\dfrac{\partial G(XR^K)}{\partial R}\right|_X - \dfrac{G(XR^K)}{R^K}\right\} & \text{for } G_n(x_n) \text{ increasing} \\ -\operatorname{sgn}\left\{\left.\dfrac{\partial G(XR^K)}{\partial R}\right|_X - \dfrac{G(XR^K)}{R^K}\right\} & \text{for } G_n(x_n) \text{ decreasing} \end{cases}$$

$\left.\dfrac{\partial G(XR^K)}{\partial R}\right|_X - \dfrac{G(XR^K)}{R^K} < 0$ follows from Proposition 5-1 a).

□

To show that $\frac{\partial}{\partial X}G^*_n(x_n) > 0$, note that, from (C-10),

$$G^*_n(x_n) = G_n(x_n) + \int_{R^*_{(n+1)\ell}}^{R^*_{(n+1)h}} (G^*_{n+1}(XR)-c)\cdot p(R|R_n)dR$$

where $p_n(R|R_n)$ is the conditional density function of $\frac{x_{n+1}}{X}$, given by

$$p(R|R_n) = Xp_n(x|x_n)$$

Now $G_n(x_n)$ is increasing in X by Lemma C-5, and from (C-8), $\dfrac{\partial Xp_n(x|x_n)}{\partial X} = 0$, so we only need to show that, as X increases, $R^*_{n\ell}$ and R^*_{nh} diverge. This follows from substituting $G^*_n(XR^*_{n\ell})$ and $G^*_n(XR^*_{nh})$ for $G_n(XR^K)$ in (C-18).

□

APPENDIX D: PROOFS FROM CHAPTER 7

Derivation of Monopoly/Inefficiency Costs.

A. Average Total Search Cost (ASE).

—The average unit search cost of those consumers for whom $x_k \le x \le x_{k+1}$ is

$$\frac{x_{k+1} + x_k}{2}.$$

—Their expected number of searches

$$= \frac{k}{n} + 2\left(\frac{k}{n}\right)\left(\frac{n-k}{n}\right) + 3\left(\frac{k}{n}\right)\left(\frac{n-k}{n}\right)^2 \ldots$$

$$= (1-\beta)(1+2\beta+3\beta^2+4\beta^3\ldots) \qquad \text{where } \beta = \frac{n-k}{n}$$

$$= (1-\beta)\sum_{i=1}^{\infty} \frac{\partial}{\partial\beta}(\beta^i)$$

$$= (1-\beta)\frac{\partial}{\partial\beta}\sum_{i=1}^{\infty} \beta^i$$

$$= (1-\beta)\frac{1}{(1-\beta)^2}$$

$$= \frac{n}{k}.$$

—The proportion of consumers in that group

$$= \frac{M(x_{k+1} - x_k)}{T}.$$

Therefore,

$$\text{ASE} = \sum_{k=1}^{n}\left(\frac{x_{k+1} + x_k}{2}\right)\left(\frac{n}{k}\right)\left(\frac{x_{k+1} - x_k}{T}\right) - \frac{T}{2}.$$

$\frac{T}{2}$, the average cost of the first unit of search, is subtracted off as it is required whether or not there is price dispersion and so does not represent an efficiency loss.

Using (7-2-1),

$$ASE = \frac{1}{2nT}\sum_{k=1}^{n}\left(\frac{1}{k}\right)\left(kp_{k+1} + p_k(k-1) - 2\sum_{i=1}^{k} p_i + p_k\right)\left(kp_{k+1} - p_k(k-1) - p_k\right) - \frac{T}{2}$$

$$= \frac{1}{2nT}\sum_{k=1}^{n}\left(kp_{k+1} + kp_k - 2\sum_{i=1}^{k} p_i\right)\left(p_{k+1} - p_k\right) - \frac{T}{2}$$

$$= \frac{1}{2nT}\sum_{k=1}^{n}\left(kp_{k+1}^2 - kp_k^2 - 2p_{k+1}\sum_{i=1}^{k} p_i + 2p_k\sum_{i=1}^{k} p_i\right) - \frac{T}{2}.$$

Note that

$$\sum_{k=1}^{n} kp_{k+1}^2 = \sum_{k=1}^{n} (k-1)p_k^2 + np_{n+1}^2,$$

and $$\sum_{k=1}^{n} 2p_{k+1}\sum_{i=1}^{k} p_i = 2\sum_{k=1}^{n} p_k\sum_{i=0}^{k-1} p_i + 2n\bar{p}\cdot p_{n+1},$$

and $$\sum_{k=1}^{n} 2p_k\sum_{i=1}^{k} p_i = 2\sum_{k=1}^{n} p_k^2 + 2\sum_{k=1}^{n} p_k\sum_{i=1}^{k-1} p_i,$$

so $$ASE = \frac{1}{2nT}\left(\sum_{k=1}^{n} p_k^2 + np_{n+1}^2 - 2n\bar{p}\cdot p_{n+1}\right) - \frac{T}{2}$$

$$= \frac{1}{2nT}\left(\sum_{k=1}^{n} p_k^2 - n\bar{p}^2 + nT^2\right) - \frac{T}{2}$$

(as $p_{n+1} \equiv T+\bar{p}$),

$$= \frac{1}{2T}\text{var } p.$$

In a participant equilibrium,

$$(p_j-\bar{p}) = \frac{n-1}{2n-1}(\alpha_j-\bar{\alpha})$$

$$\Rightarrow \quad \text{var } p = \frac{(n-1)^2}{(2n-1)^2}\text{var } \alpha \tag{D-1}$$

$$\Rightarrow \quad \text{ASE} = \frac{(n-1)^2}{2T(2n-1)^2}\text{var } \alpha.$$

B. Average Production Costs And Markups.

The market-share for firm j, w_j, is

$$\frac{1}{\sum_{i=1}^{n} q_i} q_j = \frac{1}{n}\left(1 + \frac{\bar{p}-p_j}{T}\right). \tag{D-2}$$

Therefore, average excess production costs

$$= \sum_{j=1}^{n} \frac{1}{n}\left(1 + \frac{\bar{p}-p_j}{T}\right)\alpha_j - \alpha_1,$$

which in a participant equilibrium

$$= \sum_{j=1}^{n} \left(\frac{1}{n} + \frac{(n-1)}{Tn(2n-1)}(\bar{\alpha}-\alpha_j)\right)\alpha_j - \alpha_1$$

$$= \bar{\alpha} + \frac{(n-1)}{T(2n-1)}\bar{\alpha}^2 - \frac{(n-1)}{Tn(2n-1)}\sum_{j=1}^{n} \alpha_j^2 - \alpha_1$$

$$= (\bar{\alpha}-\alpha_1) - \frac{(n-1)}{T(2n-1)}\text{var } \alpha.$$

And average markup

$$= \sum_{j=1}^{n} \frac{1}{n}\left(1 + \frac{\bar{p}-p_j}{T}\right)(p_j-\alpha_j),$$

which in a participant equilibrium

$$= \sum_{j=1}^{n} \left(\frac{n}{n-1}T + \frac{n}{(2n-1)}(\bar{\alpha}-\alpha_j)\right)\left(\frac{1}{n} + \frac{(n-1)}{Tn(2n-1)}(\bar{\alpha}-\alpha_j)\right)$$

$$= \frac{n}{n-1}T - \frac{n(n-1)}{(2n-1)^2 T}\bar{\alpha}^2 + \frac{(n-1)}{(2n-1)^2 T}\sum_{j=1}^{n}\alpha_j^2$$

$$= \frac{n}{n-1}T + \frac{(n-1)n}{(2n-1)^2 T}\text{var } \alpha.$$

Proof of Theorem 7-2.

a) From (7-3-7),

$$CC^n = (\bar{\alpha}^n - \alpha_1) + \frac{n}{n-1}T - \frac{(n-1)^2}{2T(2n-1)^2}\text{var}^n\alpha$$

so $$\Delta CC = \frac{1}{n}(\alpha_e - \bar{\alpha}^{n-1}) - \frac{1}{(n-1)(n-2)}T + \frac{(n-2)^2}{2T(2n-3)^2}\text{var}^{n-1}\alpha - \frac{(n-1)^2}{2T(2n-1)^2}\text{var}^n\alpha$$

Note that $\text{var}^n\alpha = \frac{1}{n}\sum(\alpha_j - \bar{\alpha}^n)^2$

$$= \frac{1}{n}\left((n-1)\text{var}^{n-1}\alpha + (\alpha_e - \bar{\alpha}^{n-1})^2 + \sum_{j=1}^{n}(\alpha_j - \bar{\alpha}^n)^2 - \sum_{j=1}^{n}(\alpha_j - \bar{\alpha}^{n-1})^2\right)$$

$$= \frac{1}{n}\left((n-1)\text{var}^{n-1}\alpha + (\alpha_e - \bar{\alpha}^{n-1})^2 - n(\bar{\alpha}^n - \bar{\alpha}^{n-1})^2\right)$$

$$= \frac{1}{n}\left((n-1)\text{var}^{n-1}\alpha + (\alpha_e - \bar{\alpha}^{n-1})^2 - \frac{1}{n}(\alpha_e - \bar{\alpha}^{n-1})^2\right)$$

$$= \frac{n-1}{n}\left(\text{var}^{n-1}\alpha + \frac{1}{n}(\alpha_e - \bar{\alpha}^{n-1})^2\right) \quad \text{(D-3)}$$

so $$\Delta CC = \frac{1}{n}(\alpha_e - \bar{\alpha}^{n-1}) - \frac{(n-1)^3}{2Tn^2(2n-1)^2}(\alpha_e - \bar{\alpha}^{n-1})^2 - \frac{1}{(n-1)(n-2)}T$$

$$+ \frac{1}{2T}\left(\frac{(n-2)^2}{(2n-3)^2} - \frac{(n-1)^3}{n(2n-1)^2}\right)\text{var}^{n-1}\alpha. \quad \text{(D-4)}$$

and $$\frac{\partial \Delta CC}{\partial \alpha_e} = \frac{1}{n} - \frac{(n-1)^3}{Tn^2(2n-1)^2}(\alpha_e - \bar{\alpha}^{n-1}). \tag{D-5}$$

Let α_e^m be the supremum of the set of α_e that give positive output. From equation (7-3-7),

$$\alpha_e^m - \frac{(n-1)\bar{\alpha}^{n-1} + \alpha_e^m}{n} = \frac{2n-1}{n-1}T$$

$$\Rightarrow \quad (\alpha_e^m - \bar{\alpha}^{n-1}) = \frac{n(2n-1)}{(n-1)^2}T. \tag{D-6}$$

Equation (D-5) is minimized when $\alpha_e = \alpha_e^m$. Using (D-6),

$$\frac{\partial \Delta CC}{\partial \alpha_e^m} = \frac{1}{n} - \frac{(n-1)}{n(2n-1)} > 0.$$

ΔCC is therefore monotonic increasing in α_e, proving the existence of the critical point. (D-6) into (D-4) gives

$$\Delta CC\Big|_{(\alpha_e = \alpha_e^m)} = \frac{3n^2-8n+3}{2(n-1)^2(n-2)}T + \frac{1}{2T}\left(\frac{(n-2)^2}{(2n-3)^2} - \frac{(n-1)^3}{n(2n-1)^2}\right)\mathrm{var}^{n-1}\alpha,$$

which is strictly positive so that $\alpha_e^c < \alpha_e^m$.

□

b) From equation (7-3-8),

$$SC^n = (\bar{\alpha}^n - \alpha_1) - \frac{(n-1)(3n-1)}{2T(2n-1)^2}\mathrm{var}^n\alpha,$$

so $$\Delta SC = \frac{(\alpha_e - \bar{\alpha}^{n-1})}{n} + \frac{(n-2)(3n-4)}{2T(2n-3)^2}\mathrm{var}^{n-1}\alpha - \frac{(n-1)(3n-1)}{2T(2n-1)^2}\mathrm{var}^n\alpha$$

$$= \frac{1}{n}(\alpha_e - \bar{\alpha}^{n-1}) - \frac{(n-1)^2(3n-1)}{2Tn^2(2n-1)^2}(\alpha_e - \bar{\alpha}^{n-1})^2$$

$$+ \frac{1}{2T}\left(\frac{(n-2)(3n-4)}{(2n-3)^2} - \frac{(n-1)^2(3n-1)}{n(2n-1)^2}\right)\text{var}^{n-1}\alpha \qquad \text{(D-7)}$$

(using (D-3).

Then, $$\frac{\partial \Delta SC}{\partial \alpha_e} = \frac{1}{n} - \frac{(n-1)^2(3n-1)}{Tn^2(2n-1)^2}(\alpha_e - \bar{\alpha}^{n-1}).$$

This is a monotonic increasing function for $\alpha_e < \bar{\alpha}^{n-1}$, while at $\alpha_e = \bar{\alpha}^{n-1}$,

$$\Delta SC = \frac{1}{2T}\left(\frac{(n-2)(3n-4)}{(2n-3)^2} - \frac{(n-1)^2(3n-1)}{n(2n-1)^2}\right)\text{var}^{n-1}\alpha > 0.$$

Therefore the result holds for $\alpha_e \le \bar{\alpha}^{n-1}$. It remains to show that

$$\alpha_e > \bar{\alpha}^{n-1} \quad \Rightarrow \quad \Delta SC > 0.$$

Note that $\partial^2 \Delta SC/\partial \alpha_e^2 < 0$ so that it is sufficient to show that $\Delta SC > 0$ at the maximum value α_e^m. (D-6) into (D-7) gives,

$$\Delta SC\Big|_{(\alpha_e = \alpha_e^m)} = \frac{1}{2(n-1)}T + \frac{1}{2T}\left(\frac{(n-2)(3n-4)}{(2n-3)^2} - \frac{(n-1)^2(3n-1)}{n(2n-1)^2}\right)\text{var}^{n-1}\alpha,$$

which is always positive.

□

c) It is sufficient to show that $\alpha_e^c > \alpha_e^s$. From (D-4) and (D-7),

$$\Delta SC - \Delta CC = \frac{-(n-1)^2}{Tn(2n-1)^2}(\alpha_e - \bar{\alpha}^{n-1})^2 + \frac{1}{(n-1)(n-2)}T + \frac{(4n^2-12n+7)(n-1)}{(2n-1)^2(2n-3)^2 T}\text{var}^{n-1}\alpha,$$

which is negative for low and high values of α_e and positive in-between.

Let α_e^o be the value of α_e such that $\Delta SC\text{-}\Delta CC = 0$ with $\alpha_e^o < \bar{\alpha}^{n-1}$.

Substituting into (D-7) gives

$$\Delta SC\big|_{(\alpha_e=\alpha_e^o)} = \frac{1}{n}(\alpha_e^o-\bar{\alpha}^{n-1}) - \frac{(3n-1)}{2n(n-1)(n-2)}T$$

$$- \left[\frac{(4n^2-12n+7)(n-1)(3n-1)}{2nT(2n-1)^2(2n-3)^2} + \frac{1}{2T}\left(\frac{(n-2)(3n-4)}{(2n-3)^2} - \frac{(n-1)^2(3n-1)}{n(2n-1)^2}\right)\right]\mathrm{var}\alpha$$

$$= \frac{1}{n}(\alpha_e^o-\bar{\alpha}^{n-1}) - \frac{(3n-1)}{2n(n-1)(n-2)}T - \frac{(n-2)}{2nT(2n-3)^2}\mathrm{var}\ \alpha < 0.$$

It then follows from b) of this theorem that $\alpha_e^o < \alpha_e^s < \bar{\alpha}^{n-1}$, and so from the definition of α_e^o, $\Delta CC\big|_{(\alpha_e=\alpha_e^s)} < 0$. Therefore, from a of this theorem,

$$\alpha_e^c > \alpha_e^s.$$

□

d) Follows directly from (D-4), (D-7) and (D-1).

□

e) *Range:* From equation (7-2-6),

$$p_{n-1}^{n-1}-p_1^{n-1} = \frac{n-2}{2n-3}(\alpha_{n-1}-\alpha_1),$$

and $$p_{n-1}^{n}-p_1^{n} = \frac{n-1}{2n-1}(\alpha_{n-1}-\alpha_1),$$

so $$\Delta(p_{n-1}-p_1) = \frac{1}{(2n-1)(2n-3)}(\alpha_{n-1}-\alpha_1) > 0.$$

Variance: $$p_j^{n-1}-\bar{p}^{n-1} = \frac{n-2}{2n-3}(\alpha_j-\bar{\alpha}^{n-1}),$$

so $$\mathrm{var}^{n-1}p_j = \frac{(n-2)^2}{(2n-3)^2}\mathrm{var}^{n-1}\alpha_j;$$

and $$p_j^n - \frac{1}{n-1}\sum_{j\neq e} p_j^n = \frac{n-1}{2n-1}(\alpha_j - \bar{\alpha}^{n-1}),$$

so the n-firm variance of the existing prices n-1 prices is

$$\frac{(n-1)^2}{(2n-1)^2}\mathrm{var}^{n-1}\alpha_j \;>\; \mathrm{var}^{n-1}p_j.$$

□

Proof of Theorem 7-4:

Let the superscript n denote variables from the participant equilibrium, and a the arbitrage equilibrium. From (7-2-6),

$$p_j^n = \frac{n}{n-1}T + \frac{n}{2n-1}\bar{\alpha}^n + \frac{n-1}{2n-1}\alpha_j$$

$$= \frac{n}{n-1}T + \frac{n}{2n-1}\left(\frac{(n-1)\bar{\alpha}^a + \alpha_e}{n}\right) + \frac{n-1}{2n-1}\alpha_j$$

$$= \frac{1}{(n-1)(2n-1)}\left((5n^2-6n+2)\phi + n(2n-1)\alpha_a + n(n-1)\alpha_1 + (n-1)^2\alpha_j\right) \quad \text{(D-8)}$$

(using (7-4-11) to substitute ϕ for T and $\bar{\alpha}^a$).

So $$p_j^a - p_j^n = \frac{1}{(n-1)(n-2)(2n-1)}\left((4n^2-5n+2)\phi + n^2\alpha_a\right) > 0, \quad \text{(D-9)}$$

$$p_a - p_e = \frac{1}{(n-1)(n-2)(2n-1)}\left(n(3n^2-4n+2)\phi + n(n^2-n+1)\alpha_a\right) > 0, \quad \text{(D-10)}$$

$$p_1^a - p_1^n = \frac{1}{(n-1)(n-2)(2n-1)}\left((n^3+3n^2-5n+2)\phi + n(2n-1)\alpha_a\right) > 0. \quad \text{(D-11)}$$

From (7-3-1), (7-3-2), (7-3-3),

$$CC = \frac{1}{2T}\mathrm{var}\ p + \sum_{j=1}^{n} p_j w_j - \alpha_1$$

$$\Rightarrow \quad \Delta CC = \frac{1}{2T}\Delta \text{var } p + \sum_{j=1}^{n} \Delta(p_j w_j), \tag{D-12}$$

and

$$SC = \frac{1}{2T}\text{var } p + \sum_{j=1}^{n} \alpha_j w_j - \alpha_1$$

$$\Rightarrow \quad \Delta SC = \frac{1}{2T}\Delta \text{var } p + \sum_{j=1}^{n} \alpha_j \Delta w_j, \tag{D-13}$$

(where $\Delta CC = CC^a - CC^n$ etc).

Note that all prices have risen by $p_j^a - p_j^n$, given by equation (D-9), with additional rises in p_a, p_1,

$$\Delta p_a = \frac{1}{(n-2)(2n-1)}\left((n-1)(3n-2)\phi + n(n-1)\alpha_a\right) \tag{D-14}$$

$$\Delta p_1 = \frac{1}{(n-2)(2n-1)}\left(n^2\phi + n\alpha_a\right). \tag{D-15}$$

A constant rise in all prices will affect neither var p nor the weights w_j so, with the qualification that $p_j^a - p_j^n$ has to be added to the value derived for ΔCC, the remainder of the derivation will reason as if Δp_a and Δp_1 were the only changes in price.

The 3 components of (D-12) and (D-13) are calculated in turn.

A. $\frac{1}{2T}\Delta var\ p$.

$$n\text{var } p = \sum_{j=1}^{n} p_j^2 - n\bar{p}^2,$$

so

$$n\Delta\text{var } p = \Delta p_1^2 + \Delta p_a^2 - n\Delta\bar{p}^2$$

$$= \Delta p_1(2p_1 + \Delta p_1) + \Delta p_a(2p_e + \Delta p_a) - n\Delta p(2\bar{p} + \Delta\bar{p}),$$

where p_1, p_e, $\bar{p}$ are the initial, participant equilibrium values of these variables.

Now $\Delta\bar{p} = \dfrac{\Delta p_1 + \Delta p_a}{n}$,

so $n\Delta \text{var } p = 2\Delta p_1(p_1-\bar{p}) + 2\Delta p_a(p_e-\bar{p}) + \Delta p_1^2 + \Delta p_a^2 - \frac{1}{n}(\Delta p_1 + \Delta p_a)^2$. (D-16)

From (7-2-6), $(p_1-\bar{p}) = \frac{n-1}{2n-1}(\alpha_1 - \bar{\alpha}^n)$,

and $(p_e-\bar{p}) = \frac{n-1}{2n-1}(\alpha_e - \bar{\alpha}^n)$,

but $\bar{\alpha}^n = \dfrac{(n-1)\bar{\alpha}^a + \alpha_e}{n}$,

so $(p_1-\bar{p}) = \dfrac{n-1}{n(2n-1)}\Big((n-1)(\alpha_1 - \bar{\alpha}) - \alpha_a\Big)$, (D-17)

and $(p_e-\bar{p}) = \dfrac{n-1}{n(2n-1)}\Big((n-1)(\alpha_1 - \bar{\alpha}) + (n-1)\alpha_a\Big)$ (D-18)

(dropping the superscript on $\bar{\alpha}^a$ from this point on).

Putting (D-14),(D-15),(D-17),(D-18), into (D-16) gives

$$\frac{1}{2T}\Delta \text{var } p = \frac{n-1}{2Tn^2(n-2)(2n-1)^2}\Big((10n^3-16n^2+9n-2)\phi^2 + n^2(3n-4)\alpha_a{}^2$$

$$+ 2(n-1)(6n^2-5n+2)\alpha_a\phi - 2n^2(n-1)(\bar{\alpha}-\alpha_1)\alpha_a - 2(n-1)(4n^2-5n+2)(\bar{\alpha}-\alpha_1)\phi\Big). \quad \text{(D-19)}$$

C. $\sum\Delta(p_jw_j)$.

$$\Delta(p_jw_j) = w_1\Delta p_j + p_j\Delta w_j + \Delta p_j\Delta w_j.$$

So $\sum_{j=1}^{n}\Delta(p_jw_j) = w_1\Delta p_1 + w_a\Delta p_a + \sum_{j=1}^{n} p_j\Delta w_j + \Delta p_1\Delta w_1 + \Delta p_a\Delta w_a$. (D-20)

By (D-2),

$$w_j = \frac{1}{n}\left(1 + \frac{1}{T}(\bar{p}-p_j)\right),$$

so from (D-17) and (D-18),

$$w_1^n = \frac{1}{(2n-1)n^2T}\left((5n^2-6n+2)\phi + n(2n-1)\alpha_a\right) \tag{D-21}$$

$$w_a^n = \frac{1}{(2n-1)n^2T}\left((5n^2-6n+2)\phi + n^2\right). \tag{D-22}$$

Now $\qquad \Delta w_j = \frac{1}{nT}(\Delta\bar{p}-\Delta p_j),$

so $\qquad \Delta w_j = \frac{1}{n^2T}(\Delta p_a+\Delta p_1)$

$$= \frac{1}{n^2T(n-2)(2n-1)}\left((4n^2-5n+2)\phi + n^2\alpha_a\right), \tag{D-23}$$

$$\Delta w_1 = \frac{1}{n^2T}(\Delta p_a-(n-1)\Delta p_1)$$

$$= \frac{1}{n^2T(n-2)(2n-1)}\left(-(n^3-4n^2+5n-2)\phi\right), \tag{D-24}$$

and $\qquad \Delta w_a = \frac{1}{n^2T}(\Delta p_1-(n-1)\Delta p_a)$

$$= \frac{1}{n^2T(n-2)(2n-1)}\left(-(3n^3-9n^2+7n-2)\phi - n^2(n-2)\alpha_a\right). \tag{D-25}$$

Putting (D-14), (D-15) and (D-21)-(D-25) into (D-20) gives

$$\sum_{j=1}^{n}\Delta(p_jw_j) = \frac{1}{n^2T(n-2)(2n-1)}\Big((20n^4-46n^3+46n^2-22n+4)\phi^2 + 2n^2(n-1)^2(\bar{\alpha}-\alpha_1)\alpha_a -$$

$$n^2(2n^2-10n+6)\alpha_a{}^2 + (2n^4+20n^3-32n^2+18n-4)\phi\alpha_a + 2(n-1)^2(4n^2-5n+2)\phi(\bar{\alpha}-\alpha_1)\Big) \tag{D-26}$$

C. $\sum\alpha_j\Delta w_j$.

(D-23)-(D-25) give

$$\sum_{j=1}^{n}\alpha_j\Delta w_j = \frac{1}{Tn^2(n-2)(2n-1)}\Big((n-1)(4n^2-5n+2)(\bar{\alpha}-\alpha_1)\phi + n^2(n-1)(\bar{\alpha}-\alpha_1)\alpha_a - (3n^3-9n^2+7n-2)\alpha_a\phi - n^2(n-2)\alpha_a^2\Big). \quad \text{(D-27)}$$

Adding (D-9), (D-19) and (D-26) gives

$$\Delta CC = \frac{1}{2Tn^2(n-2)(2n-1)^2}\Big((30n^4-72n^3+71n^2-33n+6)\phi^2 + n^2(14n^2-14n+4)\phi\alpha_a + n^2(n^2+3n-2)\alpha_a^2\Big) + \frac{1}{(n-1)(n-2)(2n-1)}\Big((4n^2-5n+2)\phi + n^2\alpha_a\Big) > 0, \quad \text{(D-28)}$$

proving that consumer costs rise as a result of arbitrage.

(D-19) added to (D-27) gives

$$\Delta SC = \frac{1}{2Tn^2(n-2)(2n-1)^2}\Big((n-1)(10n^3-16n^2+9n-2)\phi^2 + 2n(4n^2-5n+2)\phi\alpha_a + 2n(n-1)(4n^2-5n+2)\phi(\bar{\alpha}-\alpha_1) + 2n^3(n-1)(\bar{\alpha}-\alpha_1)\alpha_a - n^3(n-3)\alpha_a^2\Big). \quad \text{(D-29)}$$

To complete the proof of Theorem 7-4 b), note that if $\alpha_e \le \bar{\alpha}$, then

$$\alpha_a \le (\bar{\alpha}-\alpha_1)$$

$$\Rightarrow \quad 2n^3(n-1)(\bar{\alpha}-\alpha_1)\alpha_a - n^3(n-3)\alpha_a^2 > 0$$

and so $\Delta SC > 0$. It follows that $\alpha_e^a > \bar{\alpha}$.

$$\frac{\partial\Delta SC}{\partial\alpha_a} \lesseqqgtr 0 \quad \text{as} \quad \Big(2n^3(n-1)(\bar{\alpha}-\alpha_1) - 2n^3(n-3)\alpha_a + 2n(4n^2-5n+2)\phi + 2n(4n^2-5n+2)\alpha_a\left(\frac{\partial\phi}{\partial\alpha_a}\right) + 2(n-1)(10n^3-16n^2+9n-2)\phi\left(\frac{\partial\phi}{\partial\alpha_a}\right)\Big) \lesseqqgtr 0$$

Now, $\Delta SC \le 0 \;\Rightarrow\; n^3(n-3)\alpha_a^2 > 2n(4n^2-5n+2)\phi + 2n^3(n-1)(\bar{\alpha}-\alpha_1)$, and $\dfrac{\partial\phi}{\partial\alpha_a} < 0$.

Therefore,

$$\Delta SC \le 0 \;\Rightarrow\; \frac{\partial\Delta SC}{\partial\alpha_a} < 0.$$

The monotonicity in α_a (and hence α_e) proves the remainder of the result. □

Proof of Theorem 7-5.

a) Substituting ϕ and α_a for $(\alpha_e-\bar{\alpha})$ in (D-4) and adding to (D-28) gives,

$$\Delta CC = \frac{A}{2Tn^2(n-1)^2(n-2)(2n-1)^2(2n-3)^2}$$

where
$$\begin{aligned} A = {} & (2n-3)^2\Big((n-1)(5n^5+8n^4-33n^3+32n^2-13n+2)\phi^2 \\ & \qquad - 2n^2(2n-1)(n^3-7n^2+7n-2)\phi T + n^2(2n-1)^2(3n^2-9n+4)T^2\Big) \\ & + (2n-3)^2\Big(2n^2(n-1)(2n^3+2n^2-5n+2)\phi\alpha_a + n^2(n-1)^2(4n^2-5n+2)\alpha_a^2 + 2n^4(2n-1)T\alpha_a\Big) \\ & + n(n-2)(n-1)^2(4n^4-24n^3+47n^2-35n+9)\mathrm{var}^{n-1}\alpha. \end{aligned}$$

Note that when n=3, all coefficients are positive, while the first bracketed term can be written

$$(4n^6+17n^5-104n^4+167n^3-122n^2+43n-6)\phi^2 + n^2(2n-1)^3(n-4)T^2 + \Big((n^3-7n^2+7n-2)\phi - n^2(2n-1)T\Big)^2$$

which is always positive for $n \ge 4$. □

b) Substituting ϕ and α_a for T and α_e in (D-7) and (D-29), and adding gives

$$\Delta SC = \frac{B}{2Tn^2(n-2)(2n-1)^2(2n-3)^2}$$

where

$$\begin{aligned} B = (2n-3)^2\Big[&(n-1)(10n^3-16n^2+9n-2)\phi^2 + (n-1)^3(n-2)(\bar{\alpha}-\alpha_1)^2 \\ &+ 2(n-1)(10n^3-23n^2+16n-4)\phi\alpha_a + (n-1)(4n^3-8n^2+5n-2)\alpha_a^{\ 2} \\ &+ 2(6n^4-28n^3+37n^2-20n+4)\phi(\bar{\alpha}-\alpha_1) - 2(2n^4-9n^3+11n^2-7n+2)\alpha_a(\bar{\alpha}-\alpha_1)\Big] \\ &+ n(n-2)(12n^4-56n^3+85n^2-49n+9)\mathrm{var}^{n-1}\alpha, \end{aligned} \qquad \text{(D-30)}$$

so that

$$\mathrm{sgn}\left\{\frac{\partial \Delta SC}{\partial \alpha_a}\right\} = \mathrm{sgn}\Big\{(n-1)^3(10n^3-23n^2+16n-4)\phi + (n-1)^3(4n^3-8n^2+5n-2)\alpha_a$$
$$- (n-1)^2(2n^4-9n^3+11n^2-7n+2)(\bar{\alpha}-\alpha_1)\Big\}. \qquad \text{(D-31)}$$

From the definition of ϕ

$$(n-1)^2(\bar{\alpha}-\alpha_1) = (5n^2-6n+2)\phi + (2n^2-2n+1)\alpha_a - n(2n-1)T,$$

so $$T > 0 \Rightarrow (n-1)^2(\bar{\alpha}-\alpha_1) > (5n^2-6n+2)\phi + (2n^2-2n+1)\alpha_a. \qquad \text{(D-32)}$$

Substituting (D-32) as an equality into (D-31) gives

$$\mathrm{sgn}\left\{\frac{\partial \Delta SC}{\partial \alpha_a}\right\} > n(4n^4+2n^3-12n^2+9n-2)\phi + n^4(2n-1)\alpha_a > 0.$$

This shows the existence of the critical point provided $\Delta SC < 0$ when $\alpha_a=0$. Finally, substituting α_a for $(\bar{\alpha}-\alpha_1)$ in (D-30) gives entirely positive coefficients, proving that α_a^e must be below $(\bar{\alpha}-\alpha_1)$.

□

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